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Propagation of Dipole Radiation Through Plane Parallel Layers

by

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Abstract

This paper is concerned with the problem of finding the strength of the electromagnetic field at any distance from a dipole source under certain special conditions, of which the principal one is that the transmitting medium is divided into horizontal layers above a plane earth, with ε, μ, and σ constant in each layer. Since the source is a dipole it is possible to express the field in terms of a Hertz potential function with one component only. The final statement for the field strength is in terms of complex integrals.

There is first considered the case of a multilayer contained between two semi-infinite homogeneous media. A stream of plane waves of constant amplitude and polarized so that the H-vector is parallel to the xy-plane impinges on the multilayer from below, and the field in the multilayer is obtained. In the next stage of the problem a dipole source of radiation, considered as a superposition of plane waves, is assumed to be located in the homogeneous region below the multilayer, and the field due to the dipole is obtained. In the final stage of the problem the dipole is located within the multilayer and the field is again found. As an illustration of the theory a special case is treated in detail. This case consists of a layer of finite height containing the dipole, located between the earth and a semi-infinite layer above.

The complex integral form of the final result may permit approximations in the form of asymptotic expressions which would have computational advantages over the conventional forms involving slowly converging series. While this theory depends upon the assumption of a flat earth, the curvature of the earth is taken into account by means of the modified index of refraction.

1. Introduction

The object of this report is to explain a rather general method for dealing with a certain class of radio wave propagation problems pertaining to non-homogeneous atmospheres. This method, which consists briefly in the decomposition of the antenna radiation into plane waves and the subsequent solution of a plane wave propagation problem, is by no means essentially new - it has been used by Weyl[(1)*] and others[(2)(3)(4)] to deal with special cases of the propagation problem - but in this paper we shall be concerned with its application to the solution of a more general kind of problem. The method possesses several features worth noting:

a) The class of problems with which it deals is large and includes all those in which the atmosphere is horizontally stratified in layers of finite thickness.

b) The solution is obtained in the form of one or more complex integrals which must then be evaluated by some suitable procedure such as series representation, asymptotic expansion, or reduction to a known form. This situation is to be contrasted with the one which exists when the more usual approach to the propagation problem is employed, namely, that of expansion in so-called "normal modes." (For a comprehensive description of this procedure, see a Radiation Laboratory report by Furry[(9)]. Also papers by Watson[(5)], van der Pol and Bremmer[(6)], Eckersley[(7)(8)], Pekeris[(10)], Pearcey[(11)], and Freehafer[(12)].) These normal modes are actually transcendental functions, an infinite series of which is used to represent the solution to the propagation problem. Since this series is generally very slowly convergent, the effort and labor involved in obtaining a numerical result are considerable. It is then natural to ask whether the complex integral form of solution in which our method eventuates is any easier to interpret numerically than is the normal mode form. While for reasons given below, we have not in this paper investigated the practical evaluation of our results - our solution is therefore a purely formal one thus far - it seems probable that corresponding to certain physical conditions such as particular kinds of variation of refractive index with height, particular ranges of the ratio of wavelength to distance from the antenna, etc.,

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*References cited in the introduction are listed on page 6 at the end of article 1.

our complex integral expressions would require much less labor for their interpretation than would the corresponding normal mode expressions.

c) In this paper, our method is used in conjunction with a single component or scalar Hertz potential formulation of the problem, i.e., the unknown will be a single scalar function of position and time, just as it is in the normal mode procedure. We may also make a comparison in this regard between the present paper and one which is to appear subsequently by R. K. Luneberg of this project, in which two unknown scalar functions (not the Hertz potentials, however,) are employed. The use of two scalar potential functions broadens the class of problems which can be treated in that it permits one to consider problems with arbitrary antennas; moreover, Luneberg defines these functions so that they are applicable to the situation in which the atmospheric refractive index varies continuously with the height instead of stepwise as in the present paper; Luneberg's work is therefore more general than the present paper in the two respects mentioned. On the other hand, it should be pointed out that the present paper could be generalized through the introduction of a second Hertz potential function to include antennas as arbitrary as those with which Luneberg's treatment is capable of dealing; the extension of our method to the case of continuously variable refractive index is not so immediate, however.

Because of their greater generality, Luneberg's results will in all likelihood be investigated with respect to practical interpretation before those of the present paper. It is for this reason that we have not as yet undertaken any analysis of the complex integrals obtained here, preferring to wait until the class of problems for which Luneberg's procedure gives practical results has been decided upon. The present report, therefore, is in no sense a final solution to the problems with which it deals, but rather constitutes one theoretical approach to such problems.

In this paper, as in all recent investigations of the problem of propagation of high frequency waves in non-homogeneous atmospheres, we shall suppose the surface of the earth to be flat and correct for this departure from the true situation by a now standard device known as the modified index of refraction. It is commonly assumed that the atmosphere is stratified, that is, that the index of refraction varies only with height above the surface of the earth. We then replace the true variation of the index of refraction as a function of the height above the surface of the earth by a modified index of refraction which is given by the formula

$$N(r) = \frac{rn(r)}{bn(b)} ,$$

where $n(r)$ represents the actual index, r is the distance from the earth's center, and b is any fixed value of r, usually taken to be the earth's radius a. If we now replace r by $a + z$ this formula gives the modified index of refraction as a function of z alone, z being the height above the earth's surface in a rectangular coordinate system. In this coordinate system x and y now represent distances along the flat surface, and any point on this surface corresponds to the point on the earth having the same distance and direction from the origin along a great circle route.

Physically expressed, the justification for the modified index of refraction is roughly the following: that instead of having the earth curve away from the radiation the index is changed in such a way as to make the rays curve away from the earth. The mathematical justification for the use of this modified index of refraction, which corrects approximately for the flattening of the earth, is given most clearly in a paper by Pekeris [13], wherein estimates of the validity of this approximation are also given. Essentially the approximation is valid out to ranges of the order of the radius of the earth and to heights of the order of 1,000 feet for wave lengths larger than a centimeter. If the range or height is exceeded or the wave length is decreased the approximation becomes poorer. Actual data are given by Pekeris in this report. The earth-flattening approximation is considered for purposes of ray-tracing by Freehafer.[14]

It is also necessary in the case where the earth is treated as an imperfect conductor to correct for the index of refraction in the earth. This means that in place of the relation $n = \sqrt{\varepsilon\mu}$ where ε and μ are the dielectric constant and permeability, which holds for dielectrics, one must use $n = \sqrt{\varepsilon_c\mu}$ where $\varepsilon_c = \varepsilon - \frac{i\sigma}{\omega}$. In case of imperfect conductivity, therefore, it is necessary to replace ε and σ by $\frac{r^2}{b^2\left[n(b)\right]^2}\varepsilon$ and $\frac{r^2}{b^2\left[n(b)\right]^2}\sigma$, respectively, so that the modified index of refraction results. No correction is made for μ since it is unity in practically all atmospheres near the surface of the earth and for short distances in the earth. In the theory of this paper, which presupposes a flat earth, ε and σ have these altered values wherever they appear.

Note: References 1-7 deal with homogeneous atmospheres; references 8-12 with inhomogeneous atmospheres.

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Section I
Formulation of the Propagation Problem to be Solved.

2. Description of the Problem.

We shall now describe the kind of problem to which our method is applicable. Referring to Fig. 1, the line G represents the surface of the earth, which we will assume throughout this report to be perfectly plane. The region below G,

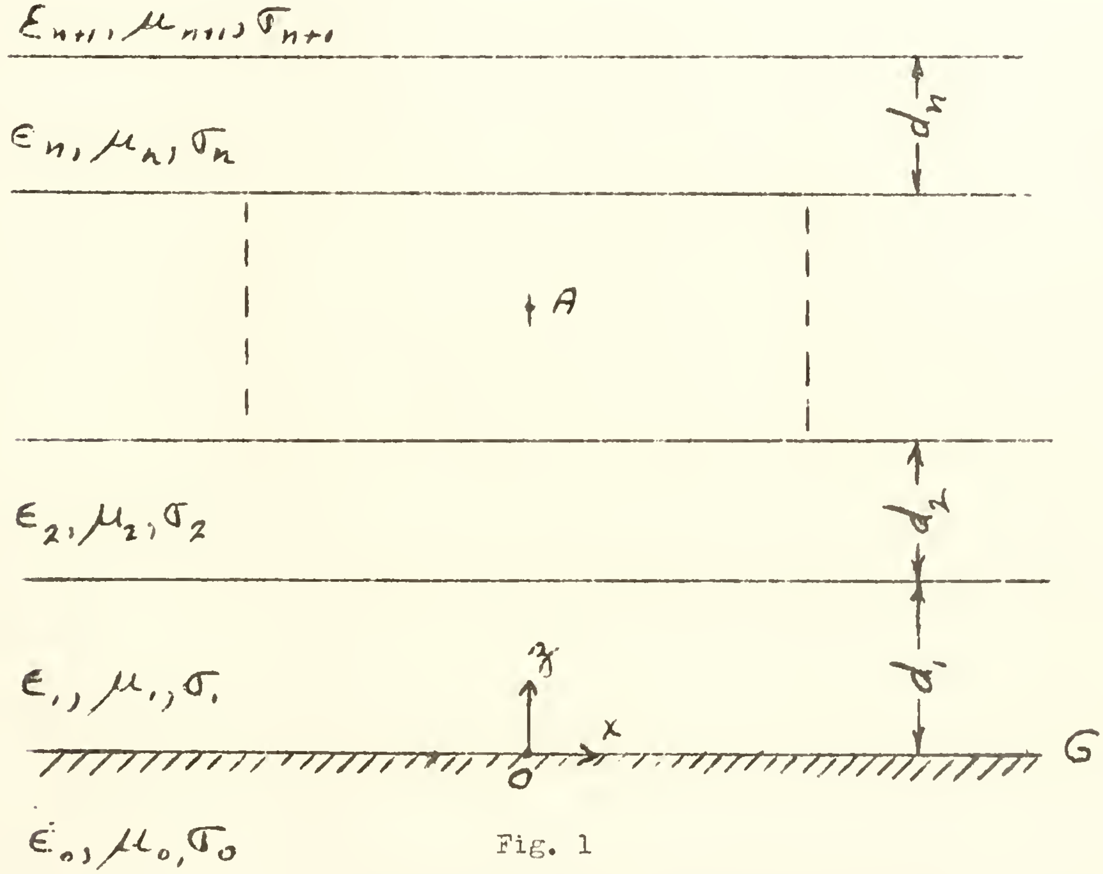

Fig. 1

Note: ϵ, μ, σ represent the dielectric constant, magnetic permeability, and electrical conductivity, respectively.

of course, constitutes the earth itself, which will generally be assumed to have a finite electrical conductivity. Above G, the atmosphere is taken to be horizontally stratified in layers of finite thickness d_1, d_2, ... d_n as shown. The boundary surfaces of each layer are horizontal planes, and within any one layer the atmosphere is perfectly homogeneous, undergoing, however, an abrupt change in its electrical

properties, e.g., its dielectric constant and electrical conductivity from one layer to the next. The number of layers n is finite, and after the n^{th} layer the atmosphere is permanently homogeneous as we move upward. The antenna A is assumed to be a vertical electric point dipole and may be located anywhere in any of the layers. The problem is to find the values of the electromagnetic field intensity vectors, $\overline{E}$ and $\overline{H}$, at any point above or on the earth. Before proceeding to the solution of this problem, however, we must first formulate it somewhat more simply and precisely.

As stated above, this propagation problem is a vector problem, i.e., we seek the field vectors $\overline{E}$ and $\overline{H}$ as functions of position (the time variation is assumed harmonic throughout), and since the vector unknowns are two in number, in general we have to find the six scalar functions E_x, E_y, E_z and H_x, H_y, H_z, i.e., the six components of E and H in the directions of the x,y,z-axes shown in Fig. 1 (the y-axis points into the paper.) If now we could find some property of our propagation problem which would enable us to reformulate it so that, instead of having to find six scalar functions we need find only one in order to solve our problem, we would obviously effect an immense simplification. We desire, in other words, a single scalar potential, from which both of the vector solutions, $\overline{E}$ and $\overline{H}$, can be derived. For the kind of problem that is represented by Fig. 1 such a scalar potential exists, and in what follows we indicate how the problem is formulated in terms of it and how the field vectors are derived from it.

3. Preliminary Formulation as a Scalar Problem.

First of all, let us consider the field distribution, $\overline{E}^*$ and $\overline{H}^*$, to which the antenna A would give rise if it were located alone in a homogeneous medium of infinite extent in all directions and characterized by the electromagnetic constants ε_j, μ_j, σ_j of the j^{th} layer of Fig. 1, this layer being the one in which the antenna of our general propagation problem is assumed to be situated. For an electric dipole in such an infinite homogeneous medium, it is known[15] that the $\overline{H}^*$ vector has no component in the antenna direction, i.e., with the orientation shown in Fig. 1, $H_z^* = 0$ everywhere. Moreover - and this is the important point - if the earth and the stratified atmosphere are now restored so that the "homogeneous

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(15) Stratton, J. A. : Electromagnetic Theory, pp. 434,435.

medium field vectors" $\bar{E}^*$ and $\bar{H}^*$ are changed to the vector distributions $\bar{E}$ and $\bar{H}$ which constitute the final solution to our propagation problem, then this solution field is also characterized by the condition $H_z = 0$. In short, the solution to the propagation problem illustrated in Fig. 1 is a so-called "transverse magnetic" field with respect to the z or antenna direction.

The latter statement follows from the fact that the boundary surfaces of the various media are all horizontal planes perpendicular to the antenna axis and hence parallel to the vector $\bar{H}^*$. Now the Maxwell boundary conditions, which the solution vectors $\bar{E}$ and $\bar{H}$ must satisfy at the aforementioned surfaces, require only that the component of $\bar{E}$ parallel to the surfaces and the component of $\mu\bar{H}$ perpendicular to the surfaces vary in a continuous manner from one side of each surface to the other; therefore, since $\mu\bar{H}^*$ is already parallel to these surfaces, i.e., possesses no perpendicular component, it is evident that the boundary condition on $\mu\bar{H}$ may be met by assuming that the perpendicular component of $\mu\bar{H}$ is equal to zero, and hence also that of $\bar{H}$. Thus the assumption that the solution field is transverse magnetic is consistent with the boundary conditions which it must satisfy.

This transverse magnetic property of the solution to our propagation problem enables us to formulate the problem in terms of a single unknown scalar potential function. We simply make use of the well known fact[16] that if in a homogeneous medium characterized by electromagnetic constants ε, μ, σ, there is an electromagnetic field which is transverse magnetic with respect to the z-direction of some xyz-coordinate system, then a scalar function of position and time $\Psi(x,y,z,t)$ exists, having the following two properties:

a) Ψ satisfies the wave equation

$$\nabla^2 \Psi - \mu\varepsilon \frac{\partial^2 \Psi}{\partial t^2} - \mu\sigma \frac{\partial \Psi}{\partial t} = 0 \qquad (1)$$

b) The components of the $\bar{E}$ and $\bar{H}$ vectors of the field are derived from Ψ by the formulas

$$E_x = \frac{\partial^2 \Psi}{\partial z \partial x}, \qquad E_y = \frac{\partial^2 \Psi}{\partial z \partial y}, \qquad E_z = -\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right), \qquad (2)$$

$$H_x = \varepsilon \frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial y}\right) + \sigma \frac{\partial \Psi}{\partial y}, \qquad H_y = -\varepsilon \frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial x}\right) - \sigma \frac{\partial \Psi}{\partial x} \qquad (3)$$

$$H_z = 0$$

[16] Stratton, op. cit., pp. 349, ff.

and from expressions (2) and (3), $\bar{E}$ and $\bar{H}$ satisfy Maxwell's equations automatically because of equation (1).

Remembering that we are considering only those fields which are harmonic in time and that we may therefore write $\Psi = \psi e^{-i\omega t}$ where $\psi = \psi(x,y,z)$, a function of position alone, we see that a) and b) become:

$$a') \quad \nabla^2 \psi + k^2 \psi = 0, \tag{4}$$

where $k^2 = \mu \epsilon \omega^2 + i\mu\sigma\omega$.

$$b') \quad E_x = \frac{\partial^2 \psi}{\partial z \partial x}, \quad E_y = \frac{\partial^2 \psi}{\partial z \partial y}, \quad E_z = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}\right) \tag{5}$$

$$H_x = \frac{k^2}{i\omega\mu} \frac{\partial \psi}{\partial y}, \quad H_y = -\frac{k^2}{i\omega\mu} \frac{\partial \psi}{\partial x}, \quad H_z = 0. \tag{6}$$

Here E_x, E_y, etc. now represent only the spatially varying parts of the field components, the field components including the time variation being given by $E_x e^{-i\omega t}$, $E_y e^{-i\omega t}$, etc.

4. Introduction of Cylindrical Coordinates.

Our propagation problem will be somewhat simplified by the use of cylindrical coordinates r, φ, z, where

$$r = \sqrt{x^2 + y^2},$$

$$\varphi = \arc\cos \frac{x}{r}, \tag{7}$$

$$z = z .$$

If the antenna of Fig. 1 is located on the z-axis of our coordinate system, then it is evident from symmetry that the resultant field is independent of the angle φ. For purposes of simplicity, therefore, we shall always take the antenna to be in this position. Hence we need concern ourselves in this section only with potential functions of the form $\psi(r, z)$, and the components of $\bar{E}$ and $\bar{H}$ with respect to the cylindrical coordinates directions are given by[17]

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*[17] Stratton, op.cit. p. 350, forms (3) & (4)

$$E_r = \frac{\partial^2 \psi}{\partial z \partial r}\ , \quad E_\varphi = 0\ , \qquad E_z = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\ , \tag{8}$$

$$H_r = 0, \quad H_\varphi = -\frac{k^2}{i\omega\mu}\frac{\partial\psi}{\partial r}\ , \quad H_z = 0 \tag{9}$$

which are analogous to equations 5 and 6.

The function ψ is known as the Hertz potential of the electromagnetic field in question. Let us now formulate the propagation problem of Fig. 1 in terms of the Hertz potential.

5. Precise Formulation of the Problem in Terms of the Hertz Potential.

The mathematically precise statement of the propagation problem is that we seek a function $\psi(r,z)$ which satisfies the following conditions:

A - Denoting the values which the function ψ takes on in the j^{th} layer of Fig. 1 by ψ_j, then as long as the antenna is not located in the j^{th} layer, we have from (4),

$$\nabla^2 \psi_j + k_j^2 \psi_j = 0 \text{ everywhere in the layer,} \tag{10}$$

where $k_j^2 = \mu_j \varepsilon_j \omega^2 + i \mu_j \sigma_j \omega$.

B - If the antenna is situated in the j^{th} layer, then ψ_j again satisfies equation (10) everywhere in the layer except at the point where the antenna is located. At this point ψ becomes infinite for the following reason. In the immediate vicinity of the antenna the field must behave like the "homogeneous medium field" of the antenna, described in article 2.† For an electric dipole located in an infinite homogeneous medium with constants ε_j, μ_j, σ_j, it can be proved that the field $\bar{E}^*$, $\bar{H}^*$, can be characterized by a Hertz potential having the form[18] ††

$$\psi_j^* = \frac{e^{ik_j R}}{R} \tag{11}$$

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† This condition on the field evidently follows from physical considerations.

18. Stratton, op. cit., pp. 431-433.

†† We are using M.K.S. units and as a consequence, ψ_j^* should really be multiplied by a factor $\frac{1}{4\pi\varepsilon_j}$. For convenience, however, we have suppressed this factor in (11) and throughout the rest of the paper.

where R is the distance from the observation point to the antenna. Our second condition on ψ_j, therefore, may be written

$$\lim_{R \to 0} \psi_j = \psi_j^* , \tag{12}$$

and from (11) we see that ψ_j must become infinite at the antenna in consequence.

C - Our third condition on ψ has to do with its behavior at large distances from the antenna. We require on physical grounds that as we proceed further and further from the antenna, the field should tend to zero, i.e.,

$$\lim_{R \to \infty} \psi = 0. \tag{13}$$

Furthermore, we require that at sufficient distances from the dipole, the field should behave like an "outgoing wave", i.e., a wave traveling away from the dipole. From a physical point of view, the necessity for this condition is apparent since its violation would imply the existence of sources of energy other than the antenna itself, which is contrary to our assumptions. The mathematical statement of this outgoing wave requirement, known as Sommerfeld's "radiation condition", is that ψ should satisfy the following:[19]

$$\lim_{R \to \infty} R\left(\frac{\partial \psi}{\partial R} - ik\psi\right) = 0. \tag{14}$$

D - Our final conditions on ψ relate to its behavior on the boundary surfaces of the various strata. This behavior is determined from the electromagnetic boundary conditions which assert that at a surface of separation of two different homogeneous media, the components of $\bar{E}$ and $\bar{H}$ tangential to the surface shall vary continuously from one side of the surface to the other[†]. Since our boundary surfaces are horizontal planes and since from (8) and (9), the only non-vanishing horizontal components of $\bar{E}$ and $\bar{H}$ are E_r and H_φ, it is to these components that we apply the conditions. Let the separating surface lie between the j^{th} and $(j + 1)^{st}$

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[19] Stratton, op. cit. pp. 485, 486.

[†] It will be noted that whereas in article 2 we employed a boundary condition requiring the continuity of the normal component of $\mu\bar{H}$, we now, for convenience, use an equivalent condition requiring the continuity of the tangential component of $\bar{H}$. The equivalence of these two conditions is proved in Stratton, op.cit.,pp.6, 34-37.

layers and let the subscripts (j) and (j + 1) denote the evaluation of quantities in the respective layers. Then on the boundary surface, we require that

$$E_{r(j)} = E_{r(j+1)} \quad , \tag{15}$$

$$H_{\varphi(j)} = H_{\varphi(j+1)} \tag{16}$$

But from (8) and (9) we see that these imply

$$\frac{\partial^2 \Psi_j}{\partial z \partial r} = \frac{\partial^2 \Psi_{j+1}}{\partial z \partial r} \tag{17}$$

$$\frac{k_j^2}{i\omega\mu_j} \frac{\partial \Psi_j}{\partial r} = \frac{k_{j+1}^2}{i\omega\mu_{j+1}} \frac{\partial \Psi_{j+1}}{\partial r} \quad . \; . \; . \tag{18}$$

Equations (17) and (18) constitute the boundary conditions on Ψ at any surface separating two of the strata. They may, however, be simplified. In the first place, both (17) and (18) hold at all points of the boundary surface, which is to say that they are true for all values of r. Hence we may integrate them with respect to r, and since from (13), the function Ψ vanishes at infinity, the constant of integration in (18) is zero. Furthermore, from (14) and (13) together we have the fact that $\frac{\partial \Psi}{\partial R}$ vanishes at infinity and hence so does $\frac{\partial \Psi}{\partial z}$. Thus, the constant of integration in (17) is also zero. Secondly, we set the magnetic permeability μ equal to unity in all the layers, corresponding to the physical situation which is almost invariably encountered. The introduction of these simplifications into (17) and (18) then leads to the following:

$$\frac{\partial \Psi_j}{\partial z} = \frac{\partial \Psi_{j+1}}{\partial z} \quad , \tag{19}$$

$$k_j^2 \Psi_j = k_{j+1}^2 \Psi_{j+1} \cdot \quad (j = 0,1,2,\ldots n) \tag{20}$$

Equations (19) and (20) represent the final form of the boundary conditions which our solution function must satisfy at the separation surface of any two of the strata.

By way of recapitulation, the mathematical statement of the propagation problem of Fig. 1 is the following. We seek a function Ψ which satisfies conditions

A,B.C, and D described above, or more specifically, which satisfies equations (10), (12), (13), (14), (19), and (20). If we can find such a function, then the field vectors $\overline{E}$ and $\overline{H}$ derived from it according to the formulas (8) and (9) or (5) and (6) constitute the solution to our propagation problem.

It will be convenient to deal with this problem by dividing it into two main cases. In the first case, we consider the problem of Fig. 1 with the dipole in the homogeneous part of the atmosphere and therefore outside of the multilayer region. This case is of importance to the problem of airplane to ground communication. In the second case, we consider the problem of Fig. 1 with the dipole located anywhere within the multilayer region. Section II will be devoted to the solution of the first of these cases.and the second case will then be solved in Section III.

Section II

Solution of the Propagation Problem with the Dipole Situated outside the Multilayer Region.

The basic procedure associated with the method we shall use consists of two parts. First we shall solve a plane wave problem which is much simpler than that of the propagation from a dipole source. The solution of the problem of propagation of dipole radiation is then "built up" out of the solutions of the simpler problem. In the first part of this section we shall describe and solve the plane wave problem. The superposition principle by which the solution of the dipole problem is built up is then explained, and the solution is obtained for the case in which the dipole is situated outside the multilayer region.

6. Description of the Plane Wave Problem.

Referring to Fig. 2, O is the origin of an x,y,z-coordinate system with the positive y-direction into the paper. The negative z-region, i.e. the region of space below the xy-plane, is filled with a homogeneous medium extending indefinitely in the x-,y-, and negative z-directions and having electromagnetic constants ε_0, σ_0 (the magnetic permeability μ, is taken as unity throughout.) The positive z-region consists of n layers or slabs extending indefinitely in the x- and y-directions and having thicknesses d_1, d_2, ... d_n in the z-direction. The boundary surfaces of these layers are, of course, planes parallel to the xy-plane, and the

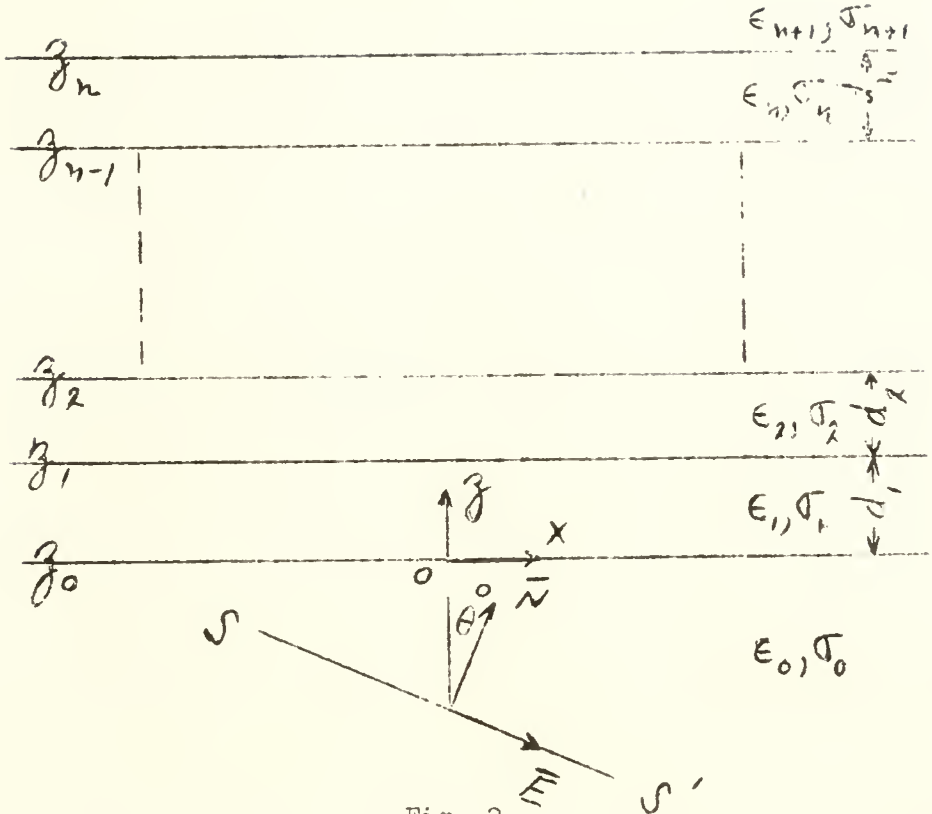

Fig. 2

electromagnetic constants of the j^{th} layer are ε_j, σ_j . Above this n-layer, or multilayer, system is another semi-infinite homogeneous medium with constants ϵ_{n+1}, σ_{n+1} . Our configuration thus comprises a multilayer system of media sandwiched between two semi-infinite, homogeneous media.

Let us now suppose that from the negative z-region a continuous stream of plane electromagnetic waves of constant amplitude and polarized so that the $\overline{H}$ vector is always parallel to the xy-plane, advances toward the multilayer, impinging upon the latter at the surface z = 0. These plane waves all have parallel wave fronts, and their common normal makes an angle θ^0 with the z-direction as shown in Fig 2, where S S' represents a typical wave front and $\overline{N}$ the normal to it. The projection of $\overline{N}$ upon the xy-plane is also presumed to make an angle φ^0 with the positive x-axis, although this angle is not shown in Fig. 2. θ^0 and φ^0 are thus the ordinary spherical polar angles of the direction $\overline{N}$, these being sufficient to specify the direction uniquely. The superscript 0 is used to denote these angles in the region $z \leq z_0$. (Note that in Fig. 2 $z_0 = 0$.)

The waves, as they pass through the multilayer, will undergo reflection and refraction at each interface so that the disturbance at any point (x,y,z), at a

given time t, will consist of the superposition of the disturbances due to all the reflected and transmitted waves passing through (x,y,z) at the time t. Since the incident waves impinge on the multilayer in a continuous and uninterrupted manner with constant amplitude, and since at a fixed point in space the incident disturbance is harmonic in time, we may assert that the disturbance anywhere within or outside the multilayer will ultimately settle down into a steady state wherein the disturbance at any point is harmonic in time and has an amplitude that is a function of position. The truth of this assertion is intuitively evident on physical grounds.

We may now ask the question: given the configuration shown in Fig. 2, what is the amplitude of the ultimate disturbance as a function of position, i.e., what are the forms of the vector functions $\overline{E}(x,y,z)$ and $\overline{H}(x,y,z)$ in the expressions $\overline{E}e^{-i\omega t}$ and $\overline{H}e^{-i\omega t}$ which constitute the ultimate disturbance under consideration? This is the plane wave problem which we set ourselves, and its importance resides in the fact that from its solution that of the dipole propagation problem can be constructed.

This simpler problem has been completely solved by R. K. Luneberg, and his solution is described in another report devoted entirely to this subject, Research Report No. 172-3. Indeed, Luneberg deals with a somewhat more general problem than the one of Fig. 2 in that he does not restrict the incident plane waves to those polarized with their $\overline{H}$ vectors parallel to the xy-plane, but admits incident plane waves of arbitrary polarization. Also - although this point is not of great significance - he allows the magnetic permeability to differ from layer to layer, while we have assumed it to be unity throughout.

Because of the arbitrary polarization of the incident plane waves in Luneberg's treatment, it is not possible for him to describe the solution of his problem in terms of a single scalar potential function. It is for this reason that we have restricted the problem of Fig. 2 to the case where the $\overline{H}$ vector of the incident wave is parallel to the xy-plane. Under such a restriction our incident field is transverse magnetic with respect to the z-direction, and hence, by the reasoning already outlined in the first section, the solution field of the plane wave problem is also transverse magnetic. But we know that this transverse magnetic property implies that there exists a scalar potential function - the Hertz potential - from which the solution vectors $\overline{E}$ and $\overline{H}$ of our plane wave problem can be derived by means of formulas (5) and (6) or (2) and (3). Thus our plane wave problem has been reduced to the finding of a single unknown scalar function, exactly as was the general propagation problem of Fig. 1.

7. Mathematical Expression for Plane Waves.

Before going further with the plane wave problem, let us write a mathematical expression which will represent the incident plane wave stream, or, for that matter, any plane wave stream. If $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ denote vectors which are constant in both space and in time, then the vector form of the incident wave system will be

$$\begin{aligned} \bar{E}^* e^{-i\omega t} &= \bar{\mathcal{E}}\, e^{ik_0(x \sin \theta^0 \cos \varphi^0 + y \sin \theta^0 \sin \varphi^0 + z \cos \theta^0) - i\omega t} \\ \bar{H}^* e^{-i\omega t} &= \bar{\mathcal{H}}\, e^{ik_0(x \sin \theta^0 \cos \varphi^0 + y \sin \theta^0 \sin \varphi^0 + z \cos \theta^0) - i\omega t} \end{aligned} \qquad (21)$$

From the form of the exponential we see that (21) represents a disturbance whose constant phase surfaces, or wavefronts, are planes the common normal to which has a direction specified by the polar angles θ^0 and φ^0. Moreover, by suitably relating the amplitude vectors $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$, it is easy to make $\bar{E}^* e^{-i\omega t}$ and $\bar{H}^* e^{-i\omega t}$ satisfy Maxwell's equations. Thus equations (21) are of the correct form for representing the incident plane waves. We desire, however, to represent the incident wave system by means of a single scalar function, rather than by several vector functions. If we denote such a scalar function by g^*, then evidently this function must have essentially the same form as $\bar{E}^*$ and $\bar{H}^*$, i.e.,

$$g^* e^{-i\omega t} = A^0\, e^{ik_0(x \sin \theta^0 \cos \varphi^0 + y \sin \theta^0 \sin \varphi^0 + z \cos \theta^0) - i\omega t} \qquad (22)$$

where A^0 is a scalar amplitude, constant in both space and time.

Application of formulas (5) and (6) to equation (22) leads directly to expressions for $\bar{E}^*$ and $\bar{H}^*$ that are identical in form with those of (21), and which automatically satisfy Maxwell's equations. From (22) we see that the incident disturbance, even when expressed in terms of the Hertz potential still consists of a stream of plane waves. Thus, our reformulation of the plane wave problem in terms of a scalar potential function has not altered its plane wave character; this character will become more evident in what follows.

8. Solution of the Plane Wave Problem.

Let us now proceed directly to the solution of the plane wave problem. Instead of stating this problem formally, however, as was done in the case of the propagation problem in Section I, we shall anticipate matters by assuming the form of the solution function and showing that this function can be made to satisfy all

the conditions which the problem imposes upon it. For convenience, moreover, we take the incident waves to be of unit amplitude, i.e., in equation (22), $A^0 = 1$. With this assumption let us denote the solution of the plane wave problem by $g\, e^{-i\omega t}$, where g may be written

$$g = g(x,y,z;\ \theta^0, \varphi^0) \tag{23}$$

Equation (23) simply expresses the fact that the amplitude of the ultimate disturbance at any point within or outside of the multilayer depends upon the coordinates (x,y,z) of the point in question and also upon the direction angles θ^0, φ^0 of the normal to the plane wave fronts of the incident waves. The choice of a particular form of the function g will be based on the physically reasonable expectation that, when a plane wave impinges upon a multilayer system such as that of Fig. 2, the ultimate disturbance in any layer will be made up of two plane waves; a forward moving or transmitted wave and a backward moving or reflected wave. We should also expect that, if in the j^{th} layer the normal to the forward moving plane wavefront makes an angle θ^j with the positive z-direction, then in accordance with the ordinary reflection law, the angle between the normal to the reflected wavefront and the positive z-direction will be $\pi - \theta^j$. Finally, we should expect that, while the angles θ of the various wavefront normals will suffer a change through refraction as we cross the boundary surface separating two layers (i.e., $\theta^j \neq \theta^{j+1}$), the angles φ of the same normals will all be equal to φ^0. The last assumption means that the various reflected and refracted wavefront normals or rays, as we may call them, all lie in the plane of incidence of the original incident wave.

All of the foregoing assumptions will be justified a posteriori. They will now be given mathematical expression through the assertion that they are equivalent to the requirement that the ultimate disturbance have the form

$$g\, e^{-i\omega t} = A^j\, e^{ik_j(x \sin\theta^j \cos\varphi^0 + y \sin\theta^j \sin\varphi^0 + z\cos\theta^j) - i\omega t}$$

$$+ B^j\, e^{ik_j(x \sin\theta^j \cos\varphi^0 + y \sin\theta^j \sin\varphi^0 - z\cos\theta^j) - i\omega t} \tag{24}$$

$$(j = 0,1,2,\ \dots\ n+1)$$

It will easily be seen by inspection that (24) embodies all of the assumptions described above. The first term on the right represents the forward moving wave in the j^{th} layer, and the second term the corresponding reflected wave. In the semi-infinite negative z-region (j = 0) the first term on the right represents

the original incident wave for which by assumption $A^0 = 1$. In the other semi-infinite medium ($j = n + 1$) there will be no reflected wave since there is no upper boundary surface, and hence $B^{n+1} = 0$. The remaining 2n+2 unknown amplitude coefficients A^j and B^j are functions of θ^0 and φ^0, and we shall now show how they are determined by the 2n+2 boundary conditions at the various separation surfaces of our multilayer system so that $g\, e^{-i\omega t}$ will be the solution to the problem of Fig. 2.

Let us suppress the time factor in (24) and write

$$g(x,y,z;\ \theta^0,\varphi^0) = A^j\, e^{ik_j(x \sin\theta^j \cos\varphi^0 + y \sin\theta^j \sin\varphi^0 + z\cos\theta^j)} + B^j\, e^{ik_j(x \sin\theta^j \cos\varphi^0 + y \sin\theta^j \sin\varphi^0 - z\cos\theta^j)} \tag{25}$$

We shall determine the coefficients A^j and B^j so that the function g satisfies our problem. We first observe the following: Denoting by g^j the values which g takes on in the j^{th} layer there results by direct substitution in equation (4)

$$\nabla^2 g^j + k_j^2 g^j = 0. \tag{26}$$

That is, regardless of the coefficients A^j and B^j, g satisfies the wave equation (26) as required, everywhere in the j^{th} layer.

For the determination of A^j and B^j we make use of the boundary conditions at the various separation surfaces of the multilayer. These boundary conditions are, as stated previously, the continuity across the surfaces of the horizontal components of the electric and magnetic fields. By formulas (5) and (6) the horizontal components in question are:

$$E_x = \frac{\partial^2 g}{\partial z\, \partial x}, \qquad E_y = \frac{\partial^2 g}{\partial z\, \partial y},$$
$$H_x = \frac{k^2}{i\omega}\frac{\partial g}{\partial y}, \qquad H_y = -\frac{k^2}{i\omega}\frac{\partial g}{\partial x}. \tag{27}$$

Now from the assumed plane wave nature of the phenomenon, it is evident that the dependence of the amplitude of the ultimate disturbance upon position in any plane parallel to the plane of incidence will be independent of the orientation of the plane of incidence with respect to some fixed coordinate system, i.e., will be independent of φ^0. Thus the unknown coefficients A^j and B^j are functions of θ^0 alone, rather than of both θ^0 and φ^0 as was stated above. For simplicity in

dealing with our boundary conditions, therefore, we shall choose $\varphi^0 = 0$. That is, the plane of incidence is the xz-plane. Since, in this case g no longer depends on y, equations (27) become

$$E_x = \frac{\partial^2 g}{\partial z \partial x} \qquad H_y = -\frac{k^2}{i\omega}\frac{\partial g}{\partial x} \tag{28}$$

Application of the boundary conditions at the surface separating the j^{th} and $(j+1)^{st}$ layers now leads to

$$\frac{\partial^2 g^j}{\partial z \partial x} = \frac{\partial^2 g^{j+1}}{\partial z \partial x}, \tag{29}$$

$$k_j^2 \frac{\partial g^j}{\partial x} = k_{j+1}^2 \frac{\partial g^{j+1}}{\partial x}, \tag{30}$$

and integration of (29) and (30) with respect to x results finally in

$$\frac{\partial g^j}{\partial z} = \frac{\partial g^{j+1}}{\partial z}, \tag{31}$$

$$k_j^2 g^j = k_{j+1}^2 g^{j+1} \tag{32}$$

$$(j = 0,1,2 \dots n)$$

Equations (31) and (32) express the fact that the Hertz potential g, which constitutes the solution of the plane wave problem, satisfies the same boundary conditions at the various horizontal interfaces as does the Hertz potential ψ which represents the solution to the propagation problem of Fig. 1, the conditions on ψ being given by (19) and (20)†. It is to be noted, incidentally, that (31) and (32) hold for all values of φ^0, the latter having been set equal to zero only for convenience in obtaining our result.

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† In integrating (29) and (30) to give (31) and (32), we have set the constants of integration equal to zero. This procedure may be justified either by means of a physical argument, into which we shall not enter, or simply by observing that either (29) and (30), or (31) and (32), lead to the same equations (33) and (34) if account be taken of condition (35).

Applying conditions (31) and (32) to the expression (25) for g at each of the separation surfaces, we have

$$A^j\, ik_j \cos\theta^j\, e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 + z_j\cos\theta^j)}$$

$$- B^j\, ik_j \cos\theta^j\, e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 - z_j\cos\theta^j)}$$

$$= A^{j+1}\, ik_{j+1} \cos\theta^{j+1}\, e^{ik_{j+1}(x\sin\theta^{j+1}\cos\varphi^0 + y\sin\theta^{j+1}\sin\varphi^0 + z_j\cos\theta^{j+1})}$$

$$- B^{j+1}\, ik_{j+1} \cos\theta^{j+1}\, e^{ik_{j+1}(x\sin\theta^{j+1}\cos\varphi^0 + y\sin\theta^{j+1}\sin\varphi^0 - z_j\cos\theta^{j+1})} \qquad (33)$$

$$A^j\, k_j^2\, e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 + z_j\cos\theta^j)}$$

$$+ B^j\, k_j^2\, e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 - z_j\cos\theta^j)}$$

$$= A^{j+1}\, k_{j+1}^2\, e^{ik_{j+1}(x\sin\theta^{j+1}\cos\varphi^0 + y\sin\theta^{j+1}\sin\varphi^0 + z_j\cos\theta^{j+1})}$$

$$+ B^{j+1}\, k_{j+1}^2\, e^{ik_{j+1}(x\sin\theta^{j+1}\cos\varphi^0 + y\sin\theta^{j+1}\sin\varphi^0 - z_j\cos\theta^{j+1})} \qquad (34)$$

$$(j = 0, 1, 2, \dots n)$$

In these expressions z_j represents the height above the xy-plane of the surface separating the j^{th} and $(j+1)^{st}$ layers; thus $z_0 = 0$.

Equations (33) and (34) constitute the 2n+2 relations from which the 2n+2 unknowns A^j and B^j are determined. These relations may be simplified, however, in view of the fact that (33) and (34) must hold everywhere on the interface $z = z_j$, i.e., they must hold for all values of x and y when $z = z_j$. Now, two such functions of x and y as those on the left and right sides of (33) and (34) can be equal for all values of x and y if, and only if, the coefficients of x and y in the exponentials on one side are equal to the corresponding coefficients in the exponentials on the other side. Thus we must have $ik_j \sin\theta^j \cos\varphi^0 = ik_{j+1} \sin\theta^{j+1} \cos\varphi^0$ and $ik_j \sin\theta^j \sin\varphi^0 = ik_{j+1} \sin\theta^{j+1} \sin\varphi^0$, both of which reduce to

$$k_j \sin\theta^j = k_{j+1} \sin\theta^{j+1} \quad . \qquad (35)$$

Equation (35) is, of course, Snell's refraction law, which we now see must be true for plane waves of Hertz potential as well as for ordinary electromagnetic waves if the function g, given by (25), is to solve the problem of Fig. 2.

Substitution of (35) into (33) and (34) reduces these equations to the final simplified form:

$$A^j k_j \cos\theta^j e^{ik_j z_j \cos\theta^j} - B^j k_j \cos\theta^j e^{-ik_j z_j \cos\theta^j} = A^{j+1} k_{j+1} \cos\theta^{j+1} e^{ik_{j+1} z_j \cos\theta^{j+1}} - B^{j+1} k_{j+1} \cos\theta^{j+1} e^{-ik_{j+1} z_j \cos\theta^{j+1}} \tag{36}$$

$$A^j k_j^2 e^{ik_j z_j \cos\theta^j} + B^j k_j^2 e^{-ik_j z_j \cos\theta^j} = A^{j+1} k_{j+1}^2 e^{ik_{j+1} z_j \cos\theta^{j+1}} + B^{j+1} k_{j+1}^2 e^{-ik_{j+1} z_j \cos\theta^{j+1}} \tag{37}$$

$$(j = 0,1,2,\dots n)$$

Using (36) and (37) and remembering that $A^0 = 1$, $B^{n+1} = 0$, we can solve for all the unknowns A^j and B^j. By the use of (35), in which the values of k_j are known in terms of ε_j, μ_j, and σ_j, the values of θ^j can be expressed in terms of θ^0.

9. Summary of the Plane Wave Problem.

Let us summarize our results thus far. We have a function $g(x,y,z;\theta^0,\phi^0)$ defined by (25) possessing the properties:

(a) $\nabla^2 g^j + k_j^2 g^j = 0$ everywhere in the j^{th} layer.

(b) If A^j and B^j satisfy (36) and (37), then g satisfies the boundary conditions (31) and (32).

(c) In every layer g represents the sum of a refracted plane wave and a reflected plane wave. In particular, in the semi-infinite negative z-region (j=0), g comprises the sum of the original incident plane wave of unit amplitude and a reflected plane wave.

In view of (a), (b) and (c) we may say that g constitutes the solution to our plane wave problem of Fig. 2 expressed in terms of the Hertz potential. We see now that the physical assumptions made on pages 19 and 20 regarding the expected behavior of these Hertz potential waves were all justified. They obey the same re-

flection and refraction laws as do light waves, at least for configurations such as that of Fig. 2.

Having solved the plane wave problem, we now proceed to an explanation of the superposition principle and of the method whereby this principle is utilized so as to construct the solution of the dipole problem from those of the plane wave problem.

10. The Superposition Principle.

First let us observe that both the differential equation (26) and the boundary conditions (31) and (32) satisfied by g are linear. From this fact we may draw two conclusions:

(a) If g_1 and g_2 are two different functions satisfying the differential equation and the boundary conditions, then $g_1 + g_2$ also satisfies the differential equation and the boundary conditions.

(b) If g is any function solving the plane wave problem with incident wave of unit amplitude, then $A^0 g$ solves the plane wave problem with incident wave of amplitude A^0.

It is clear that we may combine (a) and (b) to arrive at the following more general conclusion. Referring to Fig. 2, let there be incident upon the surface $z = 0$ of the multilayer N plane waves having amplitudes A_1^0, A_2^0, ... A_N^0 and corresponding propagation directions given by the pairs of polar angles $(\theta_1^0, \varphi_1^0)$, $(\theta_2^0, \varphi_2^0)$, ...$(\theta_N^0, \varphi_N^0)$. Then we may say that the form of the ultimate disturbance when these N plane waves impinge on the lower surface of the multilayer from the negative z-region is

$$\psi = A_1^0\, g(x,y,z;\ \theta_1^0, \varphi_1^0) + A_2^0\, g(x,y,z;\ \theta_2^0, \varphi_2^0) + \dots A_N^0\, g(x,y,z;\ \theta_N^0, \varphi_N^0)\ . \qquad (38)$$

That is, the ultimate disturbance is simply the algebraic sum of the separate ultimate disturbances due to each of the incident plane waves acting separately; in short, superposition is valid.

The significance of this result is apparent. It means that having solved the plane wave problem in which the incident field is a single plane wave of unit amplitude, we can immediately write down the solution to the more general problem where the incident field is any field which can be expressed as the sum of plane waves of arbitrary amplitudes and directions of propagation. It is necessary to know only the function $g(x,y,z;\ \theta^0, \varphi^0)$, i.e., to know the coefficients A^j and B^j of equation (25).

With the validity of the superposition principle established for the case where the incident field is representable as the sum of only a finite number of plane waves, it is natural to ask whether the principle will still be valid for the case of a more general type of incident field. In particular, we shall be interested in incident fields in which the finite sum of plane waves is replaced by a certain kind of definite integral. Deferring considerations of the validity of such a replacement for the moment, let us see how this transition is effected and what its consequences will be should superposition still prove to be valid.

We first write the expression for the incident field consisting of a finite number of plane waves, i.e.,

$$\Psi^* = \sum_{s=1}^{N} A_s^0 \, e^{ik_0(x \sin \theta_s^0 \cos \varphi_s^0 + y \sin \theta_s^0 \sin \varphi_s^0 + z \cos \theta_s^0)} \tag{41}$$

In this expression the summation extends over all the discrete values of θ^0, φ^0, and A^0 which correspond to the integral values of s from 1 to N. If, instead of permitting these variables to take on only these discrete values, we allow θ^0 and φ^0 to vary independently and continuously over the region $\theta_a^0 \leq \theta^0 \leq \theta_b^0$, $\varphi_a^0 \leq \varphi^0 \leq \varphi_b^0$, and consider A^0 to be a known function of θ^0 and φ^0, (which will usually be analytic), the expression analogous to (41) is

$$\Psi^* = \int_{\varphi_a^0}^{\varphi_b^0} d\varphi^0 \int_{\theta_a^0}^{\theta_b^0} A(\theta^0, \varphi^0)\, e^{ik_0(x \sin \theta^0 \cos \varphi^0 + y \sin \theta^0 \sin \varphi^0 + z \cos \theta^0)} \, d\theta^0 \tag{42}$$

The incident field of (42) requires the sum of a continuous infinity of plane waves.

Referring once more to Fig. 2, suppose that incident upon the lower surface of the multilayer is a field ψ^* of the type given by (42). If we assume that superposition is still valid in this case, then the ultimate disturbance ψ, by analogy with (38), would be

$$\psi = \int_{\varphi_a^0}^{\varphi_b^0} d\varphi^0 \int_{\theta_a^0}^{\theta_b^0} A(\theta^0, \varphi^0) \; g(x, y, z; \theta^0, \varphi^0) \; d\theta^0 \tag{43}$$

i.e., (43) would be the solution of the problem of Fig. 2 with incident field given by (42).

One further generalization will be needed. Consider the integral representation of the incident field ψ^* as given by (42). In this expression the variables of integration θ^0 and φ^0 take on all <u>real</u> values between the real limits of integration; the paths of integration of both θ^0 and φ^0 in (42) are segments along the real θ^0 and real φ^0 axes. Suppose, however, that we have an incident field ψ^* permitting an integral representation of the same form as (42) but requiring that in the course of the integration the variables θ^0 and φ^0 take on complex values as well as real ones. In other words, suppose that in (42) the paths of integration, instead of being segments along the real θ^0 and φ^0 axes, are contours in the complex θ^0 and φ^0 planes. The incident radiation would then be

$$\psi^* = \int_{C_{\varphi_0}} d\varphi^0 \int_{C_{\theta^0}} A(\theta^0, \varphi^0) e^{ik_0(x \sin\theta^0 \cos\varphi^0 + y \sin\theta^0 \sin\varphi^0 + z\cos\theta^0)} d\theta^0 \tag{44}$$

where C_{θ^0} and C_{φ^0} are the paths of integration of the complex variables θ^0 and φ^0 respectively, Fig. 3, the lengths of C_{θ^0} and C_{φ^0} being either finite or infinite.

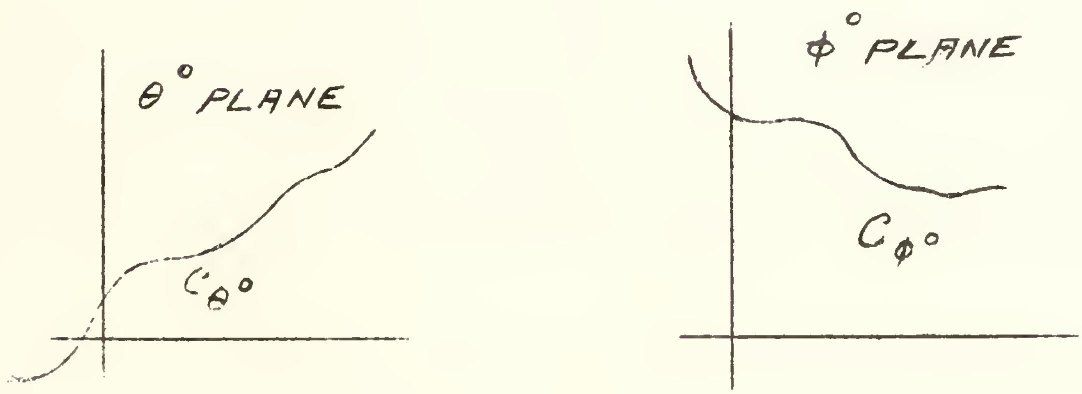

Fig. 3

We now ask the question: If in the configuration of Fig. 2 the radiation incident upon the multilayer from the negative z-region is given by (44), what is the ultimate disturbance, Ψ? In order to answer this question, let us first observe that the incident field Ψ^* in (44) is no longer represented as the superposition of ordinary plane waves since the exponential form in the integrand of (44) now contains complex angles θ^0 and φ^0 instead of real geometrical ones, and these render impossible any simple physical interpretation of the exponential as a plane wave. Mathematically speaking, however, the situation is the same as for the real case. If, in Fig. 2, we go back to the original problem, where a single plane wave of unit amplitude is incident upon the multilayer, and replace this problem by the one in which a single wave of the type $e^{ik_0(x \sin\theta^0 \cos\varphi^0 + y \sin\theta^0 \sin\varphi^0 + z \cos\theta^0)}$ is incident upon the multilayer, θ^0 and φ^0 now being complex variables, we shall find that the solution of this problem is identical in form with the solution obtained when θ^0 and φ^0 were real. The new solution will still be given by $g(x,y,z;\, \theta^0, \varphi^0)$ of (25) except that θ^0 and φ^0 have the appropriate complex values. It will still satisfy the same differential equation and the same boundary conditions as it did when θ^0 and φ^0 were real. In fact, all the mathematical properties of $g(x,y,z;\, \theta^0, \varphi^0)$, necessary for the validity of the superposition principle are possessed by this function when θ^0 and φ^0 are complex. Even

Snell's law, equation (35), carries over to the complex case, although it has no apparent geometrical significance, since the reasoning used to establish it is still valid.

In view of the foregoing, we should expect that if the generalization of the original superposition principle to incident fields representable by the real integral (42) is justified, then its generalization to the case where the incident field is expressed by the complex integral (44) is also justified. If this expectation is correct, we may summarize our conclusions by saying that when the incident field in Fig. 2 is given by (44) the solution field, or ultimate disturbance Ψ, is

$$\Psi = \int_{C_{\varphi^0}} d\varphi^0 \int_{C_{\theta^0}} A(\theta^0, \varphi^0)\, g(x,y,z;\theta^0\varphi^0)\, d\theta^0 . \qquad (45)$$

It is clear that the foregoing discussion is contingent on the validity of the superposition principle in the case represented by equation (44). While we have shown rigorously that this principle is valid when the incident wave consists of a finite number of real or complex plane waves, we have not as yet demonstrated mathematically that its generalization to an incident wave given by an integral such as that in (44) is justified. This demonstration offers no particular difficulty as long as the contours C_{θ^0} and C_{φ^0} are finite in length and pass through none of the singularities of the integrand in (45). Under these conditions, the possibility of the integral (45) or its derivatives failing to exist is precluded. If, however, it should happen, as it will in the case to be discussed shortly, that one of the contours, say C_{θ^0}, must go to infinity in some direction in order for (44) to represent the incident field Ψ^*, then (45) becomes an infinite integral and the question of its convergence and that of its derivatives becomes a rather elaborate mathematical problem, at least when we are dealing with a multilayer system such as that of Fig. 2. Unfortunately, so far as is known, this problem has not been treated in the case of the multilayer. Hence, we shall confine ourselves to two objectives. (1) We shall show how the representations (44) and (45), if their use is justified, lead to the solution of the propagation problem when the source is a dipole situated outside the multilayer. (2) We shall make more explicit the particular mathematical difficulties which stand in the way of justifying this generalization of the superposition principle.

11. Solution of the Propagation Problem with the Dipole outside the Multilayer.

For convenience we shall locate the electric dipole in the homogeneous region below the multilayer, i.e., in the negative z-region of Fig. 2. It is assumed to be vertically oriented and, without loss of generality, we may suppose it to be placed at $z = -h$. The radiation from this dipole impinges upon the multilayer, and our problem is to find the ultimate disturbance ψ anywhere within or without the multilayer.

In order to solve this problem by the use of the generalized superposition principle, we must first write down a representation of the type (44) for the incident radiation ψ^* coming from the dipole. This incident or primary radiation is simply the field to which the dipole would give rise if located alone in a medium of constant k_0 extending indefinitely in all directions. In the first section of this report it has already been stated that this field is given by

$$\psi^* = \frac{e^{ik_0R}}{R} \quad . \tag{46}$$

where R is the distance from the observation point to the point (0,0,-h) where the dipole is located. Equation (46) therefore gives the incident radiation. It can now be proved that the integral representation (44) of $\frac{e^{ik_0R}}{R}$ is given by[21][22]

$$\frac{e^{ik_0R}}{R} = \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_0}{2\pi} e^{ik_0h\cos\theta^0} \sin\theta^0 e^{ik_0(x\sin\theta^0\cos\varphi^0 + y\sin\theta^0\sin\varphi^0 + z\cos\theta^0)} d\theta^0 \tag{47}$$

for $z > -h$ and by

$$\frac{e^{ik_0R}}{R} = \int_0^{2\pi} d\varphi^0 \int_{\frac{\pi}{2} + i\infty}^{\pi} \frac{ik_0}{2\pi} e^{ik_0h\cos\theta^0} \sin\theta^0 e^{ik_0(x\sin\theta^0\cos\varphi^0 + y\sin\theta^0\sin\varphi^0 + z\cos\theta^0)} d\theta^0 \tag{48}$$

for $z < -h$.

The reason for requiring one representation for ψ^* when $z > -h$ and another when $z < -h$ is the existence of the singularity in $\frac{e^{ik_0R}}{R}$ at $R = 0$.

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(21) Stratton, op. cit. pp. 577, 578.
(22) Courant-Hilbert: Methoden der Mathematischen Physik, Vol. II, p. 154.

We observe that in both representations the function A is a function of θ^0 alone, and is the same in both (47) and (48), namely

$$A(\theta^0) = \frac{ik_0}{2\pi} e^{ik_0 h \cos \theta^0} \sin \theta^0 \tag{49}$$

With regard to the paths of integration, we see that C_{φ^0} is the same in both (47) and (48) being simply the segment of the real φ^0-axis from 0 to 2π. Fig. 4 shows C_{θ^0} for both cases.

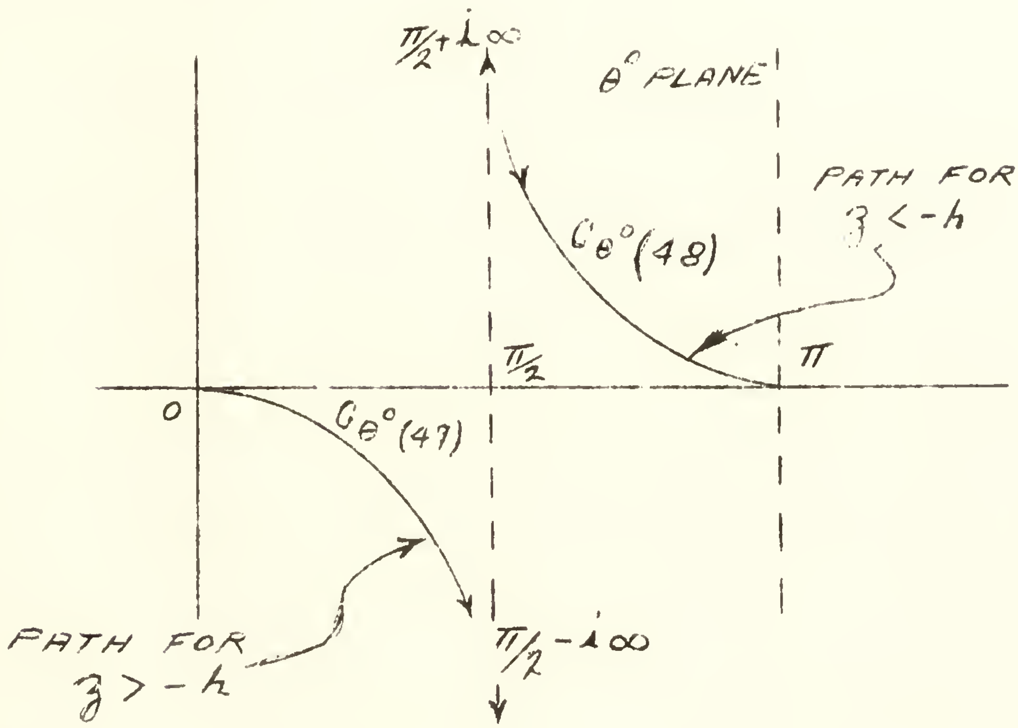

Fig. 4

We are now in a position to write down the solution of our problem. First consider the region z > -h. According to (45) and (49) the solution should be

$$\Psi = \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_0}{2\pi} e^{ik_0 h \cos \theta^0} \sin \theta^0 \, g(x,y,z;\ \theta^0, \varphi^0)\, d\theta^0 \tag{50}$$

Substituting for $g(x,y,z;\ \theta^0, \varphi^0)$ from (25) where A^j and B^j are determined by (36) and (37), equation (50) gives for the solution in the j^{th} slab of the multilayer,

$$\Psi_j = \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_0}{2\pi} e^{ik_0 h\cos\theta^0} \sin\theta^0 A^j(\theta^0) e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 + z\cos\theta^j)} d\theta^0$$

$$+ \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_0}{2\pi} e^{ik_0 h\cos\theta^0} \sin\theta^0 B^j(\theta^0) e^{ik_j(x\sin\theta^j\cos\varphi^0 + y\sin\theta^j\sin\varphi^0 - z\cos\theta^j)} d\theta^0, \tag{51}$$

for $-h < z < 0$ when $j = 0$, and for $z_{j-1} < z < z_j$ when $j = 1,2,\ldots.n$.

Note that while a variable θ^j appears in the integrand of (51), it is in reality a function of the variable of integration θ^0 through the Snell's law relation $k_j \sin\theta^j = k_0 \sin\theta^0$. If, however, we solve this equation for θ^j to obtain $\theta^j = f(\theta^0) = \arcsin(\frac{k_0}{k_j}\sin\theta^0)$, we observe that $f(\theta^0)$ is not single-valued; i.e., the Snell's law relation does not uniquely determine θ^j as a function of the variable of integration. To remove this ambiguity, the following rule is to be used: whenever the real part of the variable of integration lies between 0 and $\frac{\pi}{2}$, the real part of θ^j also lies between 0 and $\frac{\pi}{2}$; whenever the real part of the variable of integration lies between $\frac{\pi}{2}$ and π, the real part of θ^j lies between $\frac{\pi}{2}$ and π. In the present section of this report, the real part of θ^0 lies within the range 0 to $\frac{\pi}{2}$, so that only the first part of the rule is relevant. In Section III, however, we shall see that the real part of the variable of integration may lie in the range $\frac{\pi}{2}$ to π, so that the second part of the rule must be used. At no time, however, will the real part of the variable of integration lie outside the interval 0 to π. With this rule, we are assured that the integrand of (51) is a single valued function of θ^0 and φ^0.

We have therefore found the ultimate disturbance anywhere in the region $z > -h$ of Fig. 2 due to a dipole located at $(0,0,-h)$, assuming of course that the superposition principle is valid. The disturbance is given by (51) where A^j and B^j are the solutions of the system of linear equations (36) and (37). In particular,

let us write the expression for the disturbance in the region between the dipole and the multilayer, i.e., $0 > z > -h$. In this region $j = 0$ and $A^j = A^0 = 1$, Then (51) becomes

$$\Psi_0 = \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_0}{2\pi} e^{ik_0 h \cos\theta^0} \sin\theta^0 e^{ik_0(x \sin\theta^0 \cos\varphi^0 + y \sin\theta^0 \sin\varphi^0 + z\cos\theta^0)} d\theta^0 \tag{52}$$

$$+ \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_0}{2\pi} e^{ik_0 h \cos\theta^0} \sin\theta^0 B^0(\theta^0) e^{ik_0(x \sin\theta^0 \cos\varphi^0 + y \sin\theta^0 \sin\varphi^0 - z\cos\theta^0)} d\theta^0 .$$

$$(0 > z > -h)$$

If we examine (52) we see that, according to (47), the first integral in (52) is merely $\frac{e^{ik_0 R}}{R}$. Hence we may write (52):

$$\Psi_0 = \frac{e^{ik_0 R}}{R}$$
$$+ \int_0^{2\pi} d\varphi^0 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_0}{2\pi} e^{ik_0 h \cos\theta^0} \sin\theta^0 B^0(\theta^0) e^{ik_0(x\sin\theta^0 \cos\varphi^0 + y \sin\theta^0 \sin\varphi^0 - z\cos\theta^0)} d\theta^0 \tag{53}$$

The interpretation of (53) is evident. It means that in the region between the dipole and the multilayer, the disturbance consists of the primary dipole radiation which would exist if there were no multilayer plus a "reflected field" due to the multilayer and which is represented in (53) by the integral term. Physically, of course, this division of the field into a primary or incident field and a reflected field is what we should have expected. Moreover, the term $\frac{e^{ik_0 R}}{R}$ represents the primary radiation not only for $z > -h$ but also for $z < -h$, and since

we should expect that the reflected term in (53), together with its derivatives, is continuous across the z = - h plane (the $\frac{e^{ik_0 R}}{R}$ term gives the dipole singularity and there are to be no other singularities, there being no other sources) we should conclude that (53) represents the disturbance everywhere in the negative z-region, and is not restricted to the region between the dipole and the multilayer. If this expectation is correct, then (51) and (53) together constitute the complete solution of the propagation problem of the dipole situated outside the multilayer. The question of the correctness of this expectation will be found to be a part of the general question of the validity of the superposition principle represented by (44) and (45), for the case where one of the contours of integration goes to infinity. We shall now discuss this general question briefly.

12. Validity of the Generalized Superposition Principle.

At the outset let us say what we mean by the validity of the superposition principle represented by expressions (44) and (45). We mean simply that this principle is valid if any solution obtained by means of it satisfies conditions A, B, C, and D of Section I when the antenna is a single vertical electric dipole. Let us examine the solution ψ given by (51) and (53) from this point of view.

Condition A - This requires that in the j^{th} layer of Fig. 2, ψ should satisfy $\nabla^2 \psi_j + k_j^2 \psi_j = 0$. From (51) we see that ψ will satisfy this equation if it is permissible to perform the operation ∇^2 under the integral sign, since the integrand itself automatically satisfies the equation in question. Since the integrals are improper, the contour going to infinity, this interchange of the operations of differentiation and integration cannot be made without considering the convergence of the integrals. Now, in the first place we do not know that the integrals in (51) are themselves convergent. In the second place, even if these integrals are convergent, we do not know that the integrals which result from performing the ∇^2 operation under the integral sign are convergent. In order to resolve these convergence difficulties it is necessary to have a much more precise knowledge of the properties of the functions $A^j(\theta^0)$ and $B^j(\theta^0)$ than that they merely satisfy equations (36) and (37), since it is obviously on the analytic behavior of A^j and B^j that the convergence questions depend. Unfortunately, we do not as yet possess the knowledge of this behavior and therefore cannot deal rigorously with the convergence problem. This difficulty will be seen to arise in connection with the remaining conditions which ψ is supposed to satisfy.

Condition B - This condition requires that in the immediate vicinity of the dipole the function ψ behave like $\frac{e^{ik_0 R}}{R}$. According to (53), this condition would evidently mean that the integral term should remain finite and continuous at $R = 0$, a condition we have already anticipated in connection with the assertion that (53) represents the solution field everywhere in the negative z-region. Once again, however, we see that the convergence of the integral can be investigated only through a knowledge of the properties of the function $B^0(\theta^0)$.

Condition C - This states that ψ must satisfy $\lim_{R \to \infty} \psi = 0$ and $\lim_{R \to \infty} R\left[\frac{\partial \psi}{\partial R} - ik\,\psi\right] = 0$. Considerations similar to those explained in A and B evidently apply here also.

Conditions D - These are the boundary conditions (19) and (20) which ψ must satisfy. We have already seen from equations (31) and (32) that the plane waves satisfy conditions (19) and (20). The integrands of (51) therefore satisfy these conditions, and hence also the integrals, subject to the convergence considerations already discussed.

13. Conclusion to Section II.

In this section we have used the method of superposition of plane waves to solve the propagation problem of the dipole located outside a multilayer region. The solution is given by expressions (51) and (53), and the solution vectors $\bar{E}$ and $\bar{H}$ are derived from expressions (51) and (53) through the use of the formulas (5) and (6) or (8) and (9).

Section III

Solution of the Propagation Problem with the Dipole Situated within the Multilayer Region

14. Statement of Problem.

In this section we shall be concerned with the solution of the second case of the propagation problem described in the last paragraph of Section I. Referring to Fig. 5, which is merely Fig. 1 with added details, we have O as the origin of a rectangular coordinate system with the positive sense of the y-axis into the paper. G G represents the surface of the earth, which is also the xy plane. The

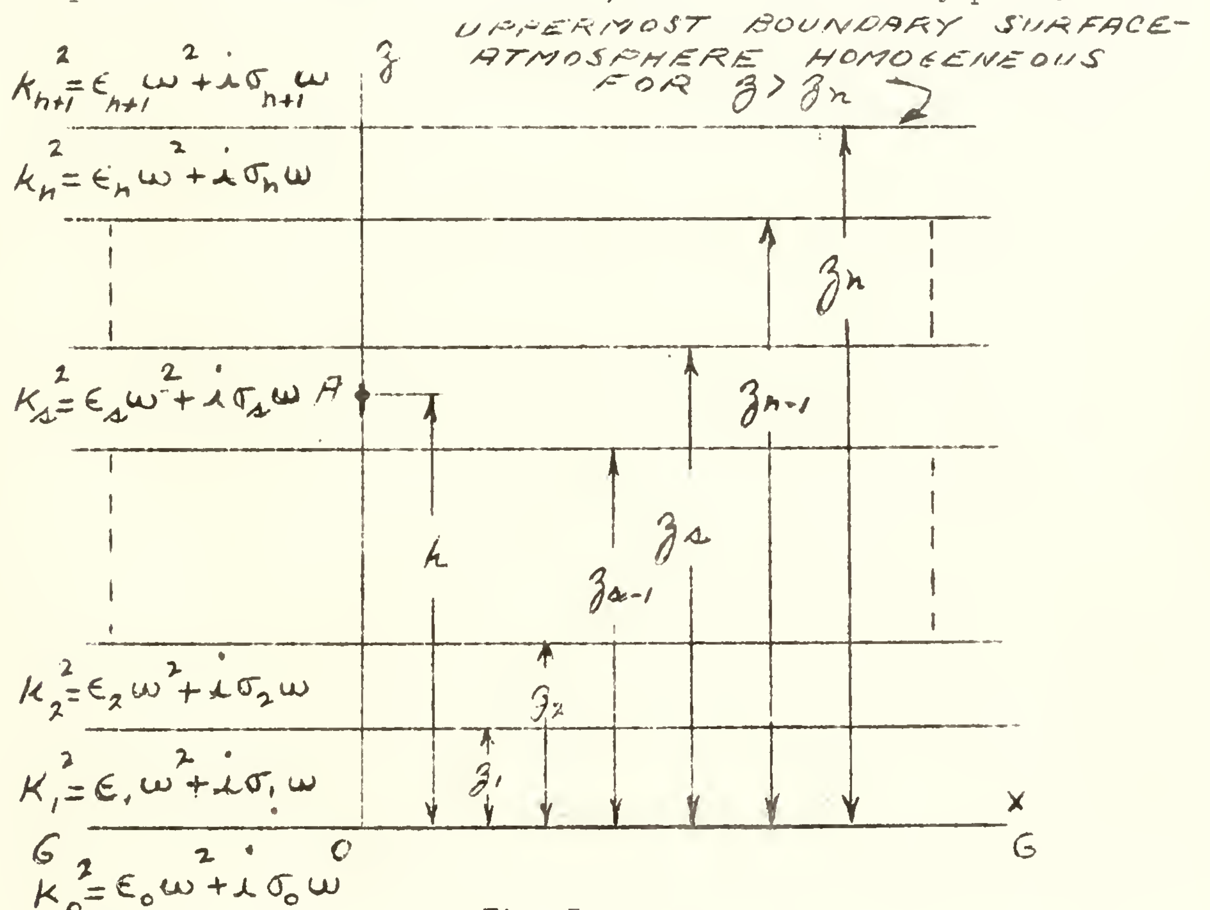

Fig. 5

vertical electric dipole of angular frequency ω is located in the s^{th} atmospheric layer at a height h above the earth and is situated on the z-axis. Our problem is to find the ultimate disturbance Ψ anywhere in any of the layers.

The difference between the present problem and that of Section II is that in the present problem the dipole is located within the multilayer region instead of being either below or above it. While we shall find that the general method developed in Section II - resolution into plane waves and superposition - is applicable to the

present case also, the new location of the dipole requires an additional procedure in order to obtain the solution.

15. Decomposition into Subsidiary Problems.

The procedure we shall employ is the following. We break up our propagation problem into two subsidiary problems. In the first of these, only the upper half of the dipole is considered as radiating; i.e., referring to Fig. 5, we suppress the primary radiation in the region $z_{s-1} \leq z < h$, considering the configuration of Fig. 5 to be unaltered otherwise. We then solve the problem of Fig. 5 under these conditions. In the second subsidiary problem we suppress the primary dipole radiation in the region $h < z \leq z_s$ and assume that only the lower half of the dipole radiates. The sum of the solutions of these two subsidiary problems evidently constitutes the solution of the problem of Fig. 5.

Each of these subsidiary problems will now be solved, by using the method of Section II with a suitable modification.

16. Solution of the First Subsidiary Problem.

In order to solve the first subsidiary problem through the use of our superposition principle, we must first find the form of the plane wave function g in any of the layers in Fig. 5. We begin with the s^{th} layer, i.e., the one containing the dipole.

The s^{th} layer.

In that part of the s^{th} layer between the dipole and the upper boundary surface of the layer, i.e., $h \leq z \leq z_s$, the plane wave function g^s evidently consists of three plane waves:

(a) A plane wave of unit amplitude moving in the same sense as the primary radiation. This will be identified as a plane wave component of the primary radiation when, in the final integration, we multiply g^s by the function A of equation (49) of Section II.

(b) A plane wave of unknown amplitude $B_1^{s\dagger}$, moving against the primary radiation and representing the effect of reflection from all the layers above the s^{th}.

† We affix the subscript 1 to the amplitudes associated with the first subsidiary problem; the solution to the latter will be called ψ_1. Similarly, we shall use the subscript 2 when we deal with the second subsidiary problem.

(c) Another plane wave moving with the primary radiation and of unknown amplitude A_1^s, representing the effect of reflection from all the layers below the s^{th}.

The function g^s can thus be written

$$g^s = (1 + A_1^s)\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s + z \cos\theta^s)} + B_1^s\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s - z \cos\theta^s)}, \tag{54}$$

for $h \leq z \leq z^s$.

In that part of the s^{th} layer between the dipole and the lower boundary surface of the layer, i.e., $z_{s-1} \leq z < h$, the primary radiation is absent by hypothesis and we are left with (b) and (c) giving

$$g = A_1^s\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s + z \cos\theta^s)} + B_1^s\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s - z \cos\theta^s)}, \tag{55}$$

for $z_{s-1} \leq z < h$.

Expressions (54) and (55) give us the form of our plane wave function g everywhere in the s^{th} layer.

<u>The j^{th} layer ($j \neq s$)</u>

In any layer other than the s^{th}, g evidently consists of two plane waves:

(a) A wave of unknown amplitude A_1^j traveling with the primary radiation.

(b) A wave of unknown amplitude B_1^j traveling against the primary radiation. Thus we have

$$g^j = A_1^j\, e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j + z \cos\theta^j)} + B_1^j\, e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j - z \cos\theta^j)} \tag{56}$$

$$(j = 0, 1, 2, \ldots s-1,\ s+1, \ldots n+1)$$

for $z_{j-1} \leq z \leq z_j$.

(It should be noted that $A_1^0 = B_1^{n+1} = 0$ since above the multilayer there will be no plane wave moving against the primary radiation, and below the multilayer, i.e., inside the earth, there will be no plane wave moving with the primary radiation.)

We have now to determine the unknown amplitudes A_1^j and B_1^j through the imposition of the boundary conditions (31) and (32) of Section II. These, we recall are

$$k_j^2 \; g^j = k_{j+1}^{\;\;2} \; g^{j+1} , \tag{57}$$

$$\frac{\partial g^j}{\partial z} = \frac{\partial g^{j+1}}{\partial z} , \tag{58}$$

$$(j = 0,1,2, \; . \; . \; . \; n)$$

Since conditions (57) and (58) are to hold at the surfaces of every layer, we now allow j to range from 0 to n including the values s - 1 and s. If we apply (57) and (58) to (54), (55), and (56) and observe that, as in Section II, Snell's law is a consequence of the boundary conditions, we get the following equations for A_1^j and B_1^j :

$$k_j^2(\delta_{sj} + A_1^j)\, e^{ik_j z_j \cos\theta^j} + k_j^2\, B_1^j\, e^{-ik_j z_j \cos\theta^j} =$$

$$k_{j+1}^2\, A_1^{j+1} e^{ik_{j+1} z_j \cos\theta^{j+1}} + k_{j+1}^2\, B_1^{j+1}\, e^{-ik_{j+1} z_j \cos\theta^{j+1}} \tag{59}$$

$$k_j \cos\theta^j(\delta_{sj} + A_1^j)\, e^{ik_j z_j \cos\theta^j} - k_j \cos\theta^j\, B_1^j\, e^{-ik_j z_j \cos\theta^j} =$$

$$k_{j+1} \cos\theta^{j+1}\, A_1^{j+1}\, e^{ik_{j+1} z_j \cos\theta^{j+1}} - k_{j+1} \cos\theta^{j+1} B_1^{j+1} e^{-ik_{j+1} z_j \cos\theta^{j+1}} \tag{60}$$

$$(j = 0,1,2,...n)$$

where $\delta_{sj} = \begin{matrix} 0 \; , \; j \neq s \\ 1 \; , \; j = s \end{matrix}$ and where $A_1^0 = B_1^{n+1} = 0$.

Equations (59) and (60) constitute 2n+2 equations for the 2n+2 unknown A_1^j and B_1^j, remembering that all the θ^js can be expressed in terms of any one of them (e.g., θ^s) through Snell's law (35).

With A_1^j and B_1^j determined by (59) and (60) we may regard the plane wave function g associated with the first subsidiary problem as known, and we now solve the first subsidiary problem by the use of the superposition method of Section II.

The solution will be given by (45). Since our dipole is now at the point (0,0,h) in the s^{th} layer, the variables of integration will be θ^s and φ^s instead of θ^0 and φ^0 (of course, $\varphi^0 = \varphi^s$). Moreover, the function $A(\theta^0)$ given by (49) becomes

$$A(\theta^s) = \frac{ik_s}{2\pi} e^{-ik_s h \cos\theta^s} \sin\theta^s . \tag{61}$$

From (45), (61), and the expressions (54), (55), and (56) for g, the solution of the first subsidiary problem is therefore (A_1^j and B_1^j being determined by 59 and 60) †

$$\Psi_1 = \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_s}{2\pi} e^{-ik_s h \cos\theta^s} \sin\theta^s (1+A_1^s) e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s + z \cos\theta^s)} d\theta^s$$

$$+ \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_s}{2\pi} e^{-ik_s h \cos\theta^s} \sin\theta^s B_1^s e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s - z \cos\theta^s)} d\theta^s, \tag{62}$$

for $h \leq z \leq z_s$.

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† The integration contour in (62) and (63) is the same as the one associated with the integral representation of the primary radiation. This follows from the superposition principle as expressed by (44) and (45) of Section II. Since, in the first subsidiary problem, only the upper half of the dipole radiates, the appropriate contour is that of (47). See Fig. 4.

$$\psi_1 = \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_s}{2\pi} e^{-ik_s h \cos\theta^s} \sin\theta^s A_1^j e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j + z \cos\theta^j)} d\theta^s$$

$$+ \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_s}{2\pi} e^{-ik_s h \cos\theta^s} \sin\theta^s B_1^j e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j - z \cos\theta^j)} d\theta^s, \tag{63}$$

for all z not in the range $h \leq z \leq z_s$.

Expressions (62) and (63) constitute the solution of our first subsidiary problem, remembering that $A_1^0 = B_1^{n+1} = 0$.

17. Solution of the Second Subsidiary Problem.

We recall that in the second subsidiary problem only the lower half of the dipole radiates, i.e., we suppress the primary radiation in the region $h < z \leq z_s$. We seek the ultimate disturbance ψ_2 under these conditions.

The procedure is exactly analogous to that used for the solution of the first subsidiary problem. We find the form of the plane wave function g in any of the layers in Fig. 5. If we still associate the A-amplitudes with plane waves moving in the same sense as does the primary radiation and the B-amplitudes with plane waves moving against the primary radiation, we see that g must have the form

$$g^s = (1 + A_2^s)\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s + z \cos\theta^s)} + B_2^s\, e^{ik_s(x \sin\theta^s \cos\varphi^s + y \sin\theta^s \sin\varphi^s - z \cos\theta^s)}, \tag{64}$$

for $z_{s-1} \leq z < h$

$$g^j = A_2^j\, e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j + z \cos\theta^j)} + B_2^j\, e^{ik_j(x \sin\theta^j \cos\varphi^j + y \sin\theta^j \sin\varphi^j - z \cos\theta^j)}, \quad (j = 0,1,2..n+1) \tag{65}$$

for all z not in the range $z_{s-1} \leq z < h$.

We note now that in (65) $A_2^{n+1} = B_2^0 = 0$, since above the multilayer there will be no plane wave moving with the primary radiation, while below the multilayer, i.e., inside the earth, there will be no plane wave moving against the primary radiation.

Applying the boundary conditions (57) and (58) to (64) and (65) for the determination of the unknown amplitudes, we evidently require (Snell's law is again a consequence of the boundary conditions)

$$k_j^2 A_2^j e^{ik_j z_j \cos\theta^j} + k_j^2 B_2^j e^{-ik_j z_j \cos\theta^j} =$$

$$k_{j+1}^2(\delta_{s-1,j} + A_2^{j+1})\, e^{ik_{j+1} z_j \cos\theta^{j+1}} + k_{j+1}^2 B_2^{j+1} e^{-ik_{j+1} z_j \cos\theta^{j+1}} \qquad (66)$$

$$k_j \cos\theta^j A_2^j e^{ik_j z_j \cos\theta^j} - k_j \cos\theta^j B_2^j e^{-ik_j z_j \cos\theta^j} =$$

$$k_{j+1}\cos\theta^{j+1}(\delta_{s-1,j} + A_2^{j+1})e^{ik_{j+1} z_j \cos\theta^{j+1}} - k_{j+1}\cos\theta^{j+1} B_2^{j+1} e^{-ik_{j+1} z_j \cos\theta^{j+1}} \qquad (67)$$

$$(j = 0,1,2,\ldots n)$$

where $\delta_{s-1,j} = \begin{matrix} 0, j \neq s-1 \\ 1, j = s-1 \end{matrix}$ and $A_2^{n+1} = B_2^0 = 0$.

Once again, we have 2n+2 equations (66) and (67) for the 2n+2 unknowns A_2^j and B_2^j, remembering that all the θ^js can be expressed in terms of θ^s through Snell's law (35).

With the amplitudes thus determined, we may write down the solution of the second subsidiary problem. We again use (45) as our basic expression. Since we are now considering that only the lower half of the dipole radiates, however, the integration contour appropriate to the solution will be that of expression (48) in Section II. This contour is also shown in Fig. 4.

The function $A(\theta^s)$ being given by (61), the function g by (64) and (65), and the amplitudes A_2^j and B_2^j being determined by (66) and (67), we have as the solution of the second subsidiary problem.

$$\Psi_2=\int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s(1+A_2^s) e^{ik_s(x\sin\theta^s\cos\varphi^s+y\sin\theta^s\sin\varphi^s+z\cos\theta^s)}\,d\theta^s$$

$$+\int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s B_2^s\, e^{ik_s(x\sin\theta^s\cos\varphi^s+y\sin\theta^s\sin\varphi^s-z\cos\theta^s)}\,d\theta^s, \tag{68}$$

for $z_{s-1} \leq z < h$.

$$\Psi_2=\int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s\, A_2^j e^{ik_j(x\sin\theta^j\cos\varphi^j+y\sin\theta^j\sin\varphi^j+z\cos\theta^j)}\,d\theta^s$$

$$+\int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s B_2^j\, e^{ik_j(x\sin\theta^j\cos\varphi^j+y\sin\theta^j\sin\varphi^j-z\cos\theta^j)}\,d\theta^s, \tag{69}$$

$$(j = 0,1,2,\ldots n+1)$$

for all z not in the range $z_{s-1} \leq z < h$.

Expressions (68) and (69) constitute the solution of the second subsidiary problem, remembering that $A_2^{n+1} = B_2^0 = 0$.

18. Conclusion of Section III.

The solution of the propagation problem of Fig. 5, i.e., the problem of the dipole within a multilayer region, is given by $\Psi_1 + \Psi_2$, where Ψ_1 and Ψ_2 are the solutions of the first and second subsidiary problems, respectively. If we carry out the addition of Ψ_1 and Ψ_2 it is easy to see that the final solution Ψ of the problem of Fig. 5 will be

$$\psi = \delta_{sj}\frac{e^{ik_sR}}{R} + \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s A_1^j e^{ik_j(x\sin\theta^j\cos\varphi^j + y\sin\theta^j\sin\varphi^j + z\cos\theta^j)} d\theta^s$$

$$+ \int_0^{2\pi} d\varphi^s \int_0^{\frac{\pi}{2}-i\infty} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s B_1^j e^{ik_j(x\sin\theta^j\cos\varphi^j + y\sin\theta^j\sin\varphi^j - z\cos\theta^j)} d\theta^s \qquad (70)$$

$$+ \int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s A_2^j e^{ik_j(x\sin\theta^j\cos\varphi^j + y\sin\theta^j\sin\varphi^j + z\cos\theta^j)} d\theta^s$$

$$+ \int_0^{2\pi} d\varphi^s \int_{\frac{\pi}{2}+i\infty}^{\pi} \frac{ik_s}{2\pi} e^{-ik_s h\cos\theta^s} \sin\theta^s B_2^j e^{ik_j(x\sin\theta^j\cos\varphi^j + y\sin\theta^j\sin\varphi^j - z\cos\theta^j)} d\theta^s$$

$$(j = 0,1,2,\ldots n+1)$$

where $\delta_{sj} = \begin{cases} 0, & j \neq s \\ 1, & j = s \end{cases}$, A_1^j and B_1^j satisfy (59) and (60), A_2^j and B_2^j satisfy (66) and (67), $A_1^0 = B_1^{n+1} = A_2^{n+1} = B_2^0 = 0$, $\varphi_j = \varphi_s$ for all j, and and R is the distance from the observation point to the point (0,0,h) where the dipole is located.† The solution vectors $\bar{E}$ and $\bar{H}$ may now be derived from ψ through relations (5) and (6) or (8) and (9) of Section I.

It should be noted that the usefulness of the solution (70) depends entirely on how successfully one evaluates the complex integrals.

† According to the rule explained in Section II, in the first two integrals of expression (70) $0 \leq \mathrm{Re}\,\theta^j \leq \frac{\pi}{2}$ while in the last two integrals, $\frac{\pi}{2} \leq \mathrm{Re}\,\theta^j \leq \pi$.

Section IV

A Special Case - n = 1.

19. Statement of Problem.

In this section we shall obtain the solution of a special case of the propagation problem of Fig. 5 in Section III. The special case is the one for which n equals 1, i.e., referring to Fig. 6, our problem is as follows:

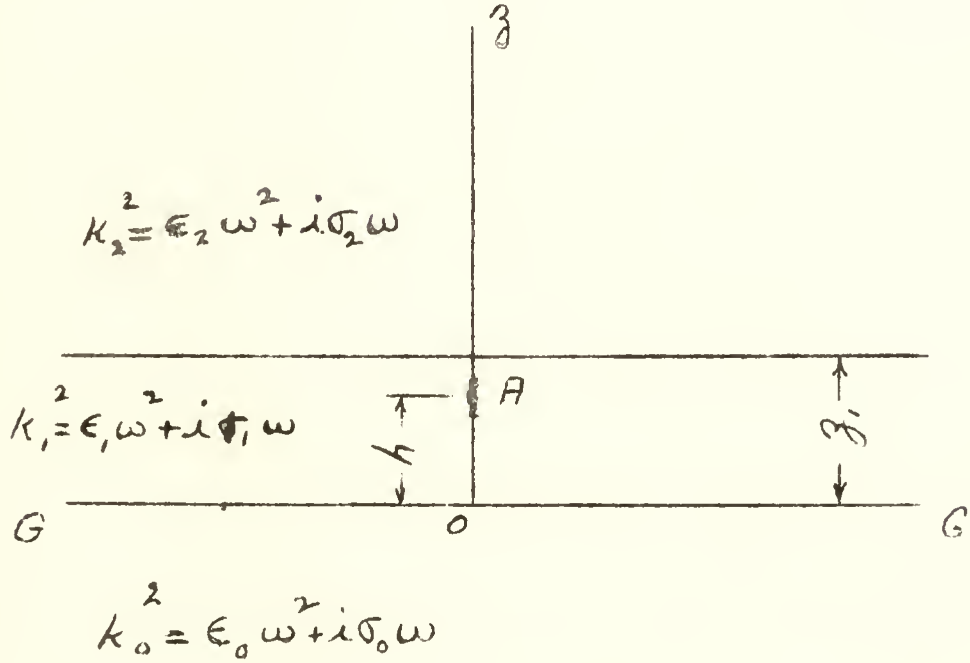

Fig. 6

The atmosphere is divided into two layers, the first of height z_1 with constants ϵ_1 and σ_1, and the second of infinite height with constants ϵ_2 and σ_2. A vertical electric dipole A of angular frequency ω is situated within the first layer at a height h above the earth. We seek the solution field ψ anywhere on or above the earth.

To solve this problem we make direct substitutions into the equations and formulas of Section III. Specifically, we solve (59) and (60) for the amplitudes associated with the first subsidiary problem, (66) and (67) for the amplitudes associated with the second subsidiary problem, and substitute these results into (70) to obtain the final solution.

We now carry out these operations.

20. Solution of the First Subsidiary Problem.

Our equations are (59) and (60) specialized for $n = 1$. Remembering that $A_1^0 = B_1^2 = 0$, these become

$$k_1^2(1 + A_1^1)e^{ik_1z_1\cos\theta^1} + k_1^2B_1^1\, e^{-ik_1z_1\cos\theta^1} = k_2^2\, A_1^2\, e^{ik_2z_1\cos\theta^2}, \qquad (71)$$

$$k_1\cos\theta^1(1 + A_1^1)e^{ik_1z_1\cos\theta^1} - k_1\cos\theta^1 B_1^1\, e^{-ik_1z_1\cos\theta^1} = k_2\cos\theta^2 A_1^2\, e^{ik_2z_1\cos\theta^2}, \qquad (72)$$

$$k_0^2\, B_1^0 = k_1^2\, A_1^1 + k_1^2\, B_1^1, \qquad (73)$$

$$-k_0\cos\theta^0\, B_1^0 = k_1\cos\theta^1\, A_1^1 - k_1\cos\theta^1\, B_1^1 \qquad (74)$$

The exponentials are absent from equations (73) and (74), because in equations (59) and (60), $z_0 = 0$.

Equations (71) through (74) constitute four equations for the four unknowns A_1^1, A_1^2, B_1^0 and B_1^1. To solve them, we proceed as follows: we multiply equation (71) by $\cos\theta^2$ and equation (72) by k_2 and subtract equation (72) from equation (71), resulting in

$$A_1^1(k_1^2\cos\theta^2 - k_1k_2\cos\theta^1)e^{ik_1z_1\cos\theta^1} + B_1^1(k_1^2\cos\theta^2 + k_1k_2\cos\theta^1)\, e^{-ik_1z_1\cos\theta^1} = (k_1k_2\cos\theta^1 - k_1^2\cos\theta^2)e^{ik_1z_1\cos\theta^1} \qquad (75)$$

We now multiply equation (73) by $\cos\theta^0$ and equation (74) by k_0, add and obtain

$$A_1^1(k_1^2\cos\theta^0 + k_1k_0\cos\theta^1) + B_1^1(k_1^2\cos\theta^0 - k_1k_0\cos\theta^1) = 0. \qquad (76)$$

Cancellation of a k_1 from both sides of equations (75) and (76) results in

$$A_1^1(k_1\cos\theta^2 - k_2\cos\theta^1)e^{ik_1z_1\cos\theta^1} + B_1^1(k_1\cos\theta^2 + k_2\cos\theta^1)e^{-ik_1z_1\cos\theta^1} = (k_2\cos\theta^1 - k_1\cos\theta^2)e^{ik_1z_1\cos\theta^1} \qquad (77)$$

$$A_1^1(k_1\cos\theta^0 + k_0\cos\theta^1) + B_1^1(k_1\cos\theta^0 - k_0\cos\theta^1) = 0. \qquad (78)$$

Equations (77) and (78) can now be solved for A_1^1 and B_1^1. Solution of equation (78) for A_1^1 gives

$$A_1^1 = \frac{k_0 \cos \theta^1 - k_1 \cos \theta^0}{k_0 \cos \theta^1 + k_1 \cos \theta^0} B_1^1 = f_0 B_1^1 , \tag{79}$$

$$\text{where} \quad f_0 = \frac{k_0 \cos \theta^1 - k_1 \cos \theta^0}{k_0 \cos \theta^1 + k_1 \cos \theta^0}$$

Substituting from equation (79) into equation (77),

$$f_0 B_1^1(k_1 \cos \theta^2 - k_2 \cos \theta^1)e^{ik_1 z_1 \cos \theta^1} + B_1^1(k_1 \cos \theta^2 + k_2 \cos \theta^1)e^{-ik_1 z_1 \cos \theta^1} = (k_2 \cos \theta^1 - k_1 \cos \theta^2)\, e^{ik_1 z_1 \cos \theta^1} , \tag{80}$$

whence we obtain

$$B_1^1 = \frac{(k_2 \cos \theta^1 - k_1 \cos \theta^2)\, e^{ik_1 z_1 \cos \theta^1}}{f_0(k_1 \cos \theta^2 - k_2 \cos \theta^1)\, e^{ik_1 z_1 \cos \theta_1} + (k_1 \cos \theta^2 + k_2 \cos \theta^1)e^{-ik_1 z_1 \cos \theta^1}} \tag{81}$$

Writing $f_1 = \dfrac{k_2 \cos \theta^1 - k_1 \cos \theta^2}{k_2 \cos \theta^1 + k_1 \cos \theta^2}$, we finally get

$$B_1^1 = \frac{f_1\, e^{2ik_1 z_1 \cos \theta^1}}{1 - f_0 f_1\, e^{2ik_1 z_1 \cos \theta^1}} . \tag{82}$$

From equations (79) and (82) we obtain

$$A_1^1 = \frac{f_0 f_1\, e^{2ik_1 z_1 \cos \theta^1}}{1 - f_0 f_1\, e^{2ik_1 z_1 \cos \theta^1}} , \tag{83}$$

and from equations (71), (82) and (83) there results

$$A_1^2 = \left(\frac{k_1}{k_2}\right)^2 \left[1 + \frac{f_0 f_1 \, e^{2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 \, e^{2ik_1 z_1 \cos\theta^1}} \right] e^{i(k_1 z_1 \cos\theta^1 - k_2 z_1 \cos\theta^2)}$$

$$+ \left(\frac{k_1}{k_2}\right)^2 \frac{f_1 \, e^{2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 \, e^{2ik_1 z_1 \cos\theta^1}} \, e^{-i(k_1 z_1 \cos\theta^1 + k_2 z_1 \cos\theta^2)} \tag{84}$$

Equations (82),(83), and (84) constitute the solution of the first subsidiary problem. We shall not bother to solve for B_1^0, since we are not especially interested in the field within the earth.

21. Solution of the Second Subsidiary Problem.

In this case our equations are (66) and (67) specialized for $n = 1$. Remembering that $A_2^2 = B_2^0 = 0$, these become

$$k_1^2 A_2^1 \, e^{ik_1 z_1 \cos\theta^1} + k_1^2 B_2^1 \, e^{-ik_1 z_1 \cos\theta^1} = k_2^2 B_2^2 \, e^{-ik_2 z_1 \cos\theta^2} , \tag{85}$$

$$k_1 \cos\theta^1 A_2^1 \, e^{ik_1 z_1 \cos\theta^1} - k_1 \cos\theta^1 B_2^1 \, e^{-ik_1 z_1 \cos\theta^1} = -k_2 \cos\theta^2 B_2^2 \, e^{-ik_2 z_1 \cos\theta^2} \tag{86}$$

$$k_0^2 A_2^0 = k_1^2 (1 + A_2^1) + k_1^2 B_2^1 , \tag{87}$$

$$k_0 \cos\theta^0 A_2^0 = k_1 \cos\theta^1 (1 + A_2^1) - k_1 \cos\theta^1 B_2^1 . \tag{88}$$

Equations (85) through (88) constitute four equations for the four unknowns A_2^0, A_2^1, B_2^1 and B_2^2 .

Multiplying equation (85) by $\cos \theta^2$ and equation (86) by k_2 and adding, we obtain

$$A_2^1(k_1^2 \cos \theta^2 + k_1 k_2 \cos \theta^1)\, e^{ik_1 z_1 \cos \theta^1} + B_2^1(k_1^2 \cos \theta^2 - k_1 k_2 \cos \theta^1) e^{-ik_1 z_1 \cos \theta^1} = 0. \tag{89}$$

Multiplication of equation (87) by $\cos \theta^0$ and of equation (88) by k_0 and subtraction of equation (88) from equation (87) results in

$$A_2^1(k_1^2 \cos \theta^0 - k_1 k_0 \cos \theta^1) + B_2^1(k_1^2 \cos \theta^0 + k_1 k_0 \cos \theta^1) = k_1 k_0 \cos \theta^1 - k_1^2 \cos \theta^0. \tag{90}$$

Cancellation of k_1 from both sides of equations (89) and (90) gives

$$A_2^1(k_1 \cos \theta^2 + k_2 \cos \theta^1)\, e^{ik_1 z_1 \cos \theta^1} + B_2^1(k_1 \cos \theta^2 - k_2 \cos \theta^1) e^{-ik_1 z_1 \cos \theta^1} = 0. \tag{91}$$

$$A_2^1(k_1 \cos \theta^0 - k_0 \cos \theta^1) + B_2^1(k_1 \cos \theta^0 + k_0 \cos \theta^1) = k_0 \cos \theta^1 - k_1 \cos \theta^0 \tag{92}$$

Equations (91) and (92) can now be solved for A_2^1 and B_2^1.

Solution of equation (91) for A_2^1 gives

$$A_2^1 = \frac{k_2 \cos \theta^1 - k_1 \cos \theta^2}{k_2 \cos \theta^1 + k_1 \cos \theta^2}\, e^{-2ik_1 z_1 \cos \theta^1} B_2^1 = f_1\, e^{-2ik_1 z_1 \cos \theta^1} B_2^1 . \tag{93}$$

Substitution of expression (93) into equation (92) gives

$$B_2^1 f_1 (k_1 \cos \theta^0 - k_0 \cos \theta^1)\, e^{-2ik_1 z_1 \cos \theta^1} + B_2^1(k_1 \cos \theta^0 + k_0 \cos \theta^1) = k_0 \cos \theta^1 - k_1 \cos \theta^0, \tag{94}$$

whence we obtain

$$B_2^1 = \frac{k_0 \cos \theta_1 - k_1 \cos \theta^0}{f_1(k_1 \cos \theta^0 - k_0 \cos \theta^1)\, e^{-2ik_1 z_1 \cos \theta^1} + k_1 \cos \theta^0 + k_0 \cos \theta^1}, \tag{95}$$

$$B_2^1 = \frac{f_0}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} \tag{96}$$

From equations (93) and (96) there results

$$A_2^1 = \frac{f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} \tag{97}$$

and finally from equations (85), (93) and (96) we have

$$B_2^2 = \left(\frac{k_1}{k_2}\right)^2 \frac{f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} e^{i(k_2 z_1 \cos\theta^2 + k_1 z_1 \cos\theta^1)}$$

$$+ \left(\frac{k_1}{k_2}\right)^2 \frac{f_0}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} e^{i(k_2 z_1 \cos\theta^2 - k_1 z_1 \cos\theta^1)} \tag{98}$$

Equations (96), (97) and (98) constitute the solution of the second subsidiary problem.

With the first and second subsidiary problems solved for the special case of the propagation problem shown in Fig. 6, we may write down the solution of this problem in accordance with expression (70) of Section III. This solution is

$$\begin{aligned}\psi = \frac{e^{ik_1R}}{R} &+ \int_0^{2\pi} d\varphi^1 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_1}{2\pi} e^{-ik_1h\cos\theta^1} \sin\theta^1 \frac{f_0f_1 e^{2ik_1z_1\cos\theta^1}}{1 - f_0f_1 e^{2ik_1z_1\cos\theta^1}} e^{ik_1(x\sin\theta^1\cos\varphi^1 + y\sin\theta^1\sin\varphi^1 + z\cos\theta^1)} d\theta^1 \\
&+ \int_0^{2\pi} d\varphi^1 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_1}{2\pi} e^{-ik_1h\cos\theta^1} \sin\theta^1 \frac{f_1 e^{2ik_1z_1\cos\theta^1}}{1 - f_0f_1 e^{2ik_1z_1\cos\theta^1}} e^{ik_1(x\sin\theta^1\cos\varphi^1 + y\sin\theta^1\sin\varphi^1 - z\cos\theta^1)} d\theta^1 \\
&+ \int_0^{2\pi} d\varphi^1 \int_{\frac{\pi}{2} + i\infty}^{\pi} \frac{ik_1}{2\pi} e^{-ik_1h\cos\theta^1} \sin\theta^1 \frac{f_0f_1 e^{-2ik_1z_1\cos\theta^1}}{1 - f_0f_1 e^{-2ik_1z_1\cos\theta^1}} e^{ik_1(x\sin\theta^1\cos\varphi^1 + y\sin\theta^1\sin\varphi^1 + z\cos\theta^1)} d\theta^1 \\
&+ \int_0^{2\pi} d\varphi^1 \int_{\frac{\pi}{2} + i\infty}^{\pi} \frac{ik_1}{2\pi} e^{-ik_1h\cos\theta^1} \sin\theta^1 \frac{f_0}{1 - f_0f_1 e^{-2ik_1z_1\cos\theta^1}} e^{ik_1(x\sin\theta^1\cos\varphi^1 + y\sin\theta^1\sin\varphi^1 - z\cos\theta^1)} d\theta^1
\end{aligned} \tag{99}$$

for $0 \leq z \leq z_1$, **while** for $z_1 \leq z$

$$\begin{aligned}\psi = \int_0^{2\pi} d\varphi^1 \int_0^{\frac{\pi}{2} - i\infty} \frac{ik_1}{2\pi} e^{-ik_1 h_1 \cos\theta^1} \sin\theta^1 \Bigg\{ \frac{k_1^2}{k_2^2} \left[1 + \frac{f_0 f_1 e^{2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 e^{2ik_1 z_1 \cos\theta^1}} \right] e^{i(k_1 z_1 \cos\theta^1 - k_2 z_1 \cos\theta^2)} \\ + \frac{k_1^2}{k_2^2} \frac{f_1 e^{2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 e^{2ik_1 z_1 \cos\theta^1}} e^{-i(k_1 z_1 \cos\theta^1 + k_2 z_1 \cos\theta^2)} \Bigg\} e^{ik_2(x \sin\theta^2 \cos\varphi^2 + \ldots + z\cos\theta^2)} d\theta^1 \\ + \int_0^{2\pi} d\varphi^1 \int_{\frac{\pi}{2} + i\infty}^{\pi} \frac{ik_1}{2\pi} e^{-ik_1 h_1 \cos\theta^1} \sin\theta^1 \Bigg\{ \frac{k_1^2}{k_2^2} \frac{f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} e^{i(k_2 z_1 \cos\theta^2 + k_1 z_1 \cos\theta^1)} \\ + \frac{k_1^2}{k_2^2} \frac{f_0}{1 - f_0 f_1 e^{-2ik_1 z_1 \cos\theta^1}} e^{i(k_2 z_1 \cos\theta^2 - k_1 z_1 \cos\theta^1)} \Bigg\} e^{ik_2(x \sin\theta^2 \cos\varphi^2 + \ldots - z\cos\theta^2)} d\theta^1 \end{aligned} \tag{100}$$